UDC 51-7

LINEAR Q-DIFFERENCE EQUATIONS OF FIRST ORDER

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Studies on q-difference equations appeared already at the beginning of the last century in intensive works especially by F H Jackson [1], R D Carmichael [2] and other authors. Unfortunately, from the years thirty up to the beginning of the eighties, only nonsignificant interest in the area were observed. Since years eighties [3], an intensive and somewhat surprising interest in the subject reappeared in many areas of mathematics and applications including mainly new difference calculus and orthogonal polynomials, q-combinatorics, q-arithmetics, integrable systems and variational q-calculus. However, though the abundance of specialized scientific publications and a relative classicality of the subject, a lack of popularized publications in the form of books accessible to a big public including under and upper graduated students is very sensitive.

Consider the q-difference equation

$$D_q y(x) = a(x) y(qx) + b(x) \qquad (1)$$

where $D_q y(x) = \dfrac{y(x) - y(qx)}{(1-q)x}, \quad 0 < q < 1.$. The corresponding homogenous equation reads

$$D_q y(x) = a(x) y(qx) \qquad (2)$$

Detailing the D_q derivative in (2), the equation reads

$$y(x) = [1 + (1-q) x a(x)] y(qx) \qquad (3)$$

Repeating the recurrence relation in (3) N times, one gets

$$y(x) = y(x_0) \prod_{t=q^{-1}x_0}^{x} [1 + (1-q) t a(t) = y(q^N x) \prod_{i=0}^{N-1} [1 + (1-q) x q^i a(q^i x)$$

If $N \to \infty$ $N ; \infty$, with $0 < q < 1$, then, $q^N \to 0$ and one obtains

$$y(x) = y(0) \prod_{i=0}^{\infty} [1 + (1-q) q^i x a(q^i x)]$$

Example. Suppose that $a(x)=\frac{q^k-1}{q-1}.\frac{1}{q^k x-1}, k\in N$. Clearly, we have the solution

$$y(x)=y(0)\prod_{i=0}^{\infty}\left[1+(1-q)q^i x a\left(q^i x\right)\right]=y(0)\prod_{0}^{k-1}\left(1-q^i x\right)=^{def}\ y(0)(x;q)_k$$

Consider next the nonhomogenous equation (1). According to the method of "variation of constants", let

$$y(x)=c(x)y_0(x) \tag{4}$$

be its solution where $y_0(x)$ is the solution of the corresponding homogenous equation (2) and $c(x)$ is an unknown function to be determined. Loading (4) in (2), and solving the obtained equation, one obtains

$$c(x)=\int_{x_0}^{x} y_0^{-1}(t)b(t)d_q t+c$$

Where $\int_a^b y(x)d_q x=\int_0^b y(x)d_q x-\int_0^a y(x)d_q x=(1-q)b\sum_{k=0}^{\infty}q^k y(bq^k)-(1-q)a\sum_{k=0}^{\infty}q^k y(aq^k)$. Hence the general solution of (2) reads

$$y(x)=y_0(x)c+\int_{x_0}^{x} y_0(x)y_0^{-1}(t)b(t)d_q t$$

with $c=y_0^{-1}(x_0)y(x_0)$ Taking $x_0=0$, we get respectively

$$c(x)=(1-q)x\sum_{i=0}^{\infty}q^i y_0^{-1}\left(q^i x\right)b\left(q^i x\right)+c$$

and

$$y(x)=y_0(x)c+(1-q)x\sum_{i=0}^{\infty}q^i y_0(x)y_0^{-1}\left(q^i x\right)b\left(q^i x\right).$$

Note that, when applied to the equation (2), the method of undetermined constants leads to the solution

$$y(x)=y_0(x)c+\int_{x_0}^{x} y_0(x)y_0^{-1}(qt)b(t)d_q t.$$

or

$$y(x)=y_0(x)c+(1-q)x\sum_{i=0}^{\infty}q^i y_0(x)y_0^{-1}\left(q^{i+1}x\right)b\left(q^i x\right)$$

for $x_0=0$.We now observe that the solutions of (1) or (2) will remain formal as long as we will not succeed to calculate the related product explicitly, a task which is far from being elementary. However, in certain situations, the coefficient $a(x)$ could suggest a particular method of resolution. When for example, $a(x)$ is a polynomial in x, we are suggested to search the solution in form of series, as show the following few simple cases:

Case 1. Equations of the form

$$D_q y(x) = ay(x), \tag{5}$$

with a, some constant. To solve such an equation, we rewrite it as

$$y(qx) = [1 + (q-1)xa]y(x). \tag{6}$$

Linear q-difference equations of first order and search the solution under the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n. \tag{7}$$

Loading (7) in (6), one obtains

$$C_n = \left(\prod_{k=1}^{n} \frac{1-q}{1-q^k} \right) a^n. \tag{8}$$

In view of the fact that $[k]_q =^{def} \prod_{k=1}^{n} \frac{1-q^k}{1-q} \to k, q \to 1,$ one can write (8) as $C_n = C_0 \frac{a^n}{[n]_q!}$,

where $[n]_q! =^{def} \prod_{k=1}^{n} \frac{1-q^k}{1-q}$ Hence the solution in (7) is a q-version of the exponential function $c_0 \exp(ax)$:

$$y_q(x) = c_0 e_q^{ax} = c_0 \sum_{n=0}^{\infty} \frac{a}{[n]_q!} x^n.$$

Case 2. Similarly, an equation of the form

$$D_q y(x) = ay(qx) \tag{9}$$

or equivalently $y(x) = [1 + (1-q)xa]y(qx)$ has a solution of the form $y_{q^{-1}}(x) = c_0 e_{q^{-1}}^{ax} = c_0 \sum_{n=0}^{\infty} \frac{a}{[n]_{q^{-1}}!} x^n$,

where $[n]_{q^{-1}}!$ is obtained from $[n]_q!$ by replacing q by q^{-1}. The functions e_q^x and $e_{q^{-1}}^x$ are clearly q-versions of the usual exponential function e^x.

Theorem 1. If

$$D_q y = a(x)y(x)$$
$$D_q z = -a(x)z(qx).$$

$$y(x_0)z(x_0)=1.$$

Then

$$y(x)z(x)=1.$$

References

1. R D Carmichael, The general theory of linear q-difference equations // Am. J. Math. (1912) №34. P 147-168.
2. Jackson H F, q-Difference equations // Am. J. Math. (1910) №32.P 305-314.

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МАТРИЦАЛЫҚ ОПЕРАТОРЛАР БІР КЛАССЫНЫҢ САЛМАҚТЫ БАҒАЛАУЛАРЫ

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Айталық, $1< p,q <\infty$, $\frac{1}{p}+\frac{1}{p'}$ $l_p = \{f = \{f\}_{i=1}^{\infty}: \sum_{i=1}^{\infty}|f_i|^p < +\infty\}$ болсын.

Берілген жұмыста

$$\|Af\|_q \le c\|f\|_p, \qquad \forall f \in l_p \tag{1}$$

теңсіздігін қарастырамыз, мұндағы A келесі түрде анықталатын матрицалық оператор:

$$(Af)_i = \sum_{j=1}^{i} a_{ij} f_j, \tag{2}$$

$(a_{i,j})$- элементтері теріс емес үшбұрышты матрица, яғни $a_{ij} \ge 0$, *егер* $i \ge j \ge 1$ және $a_{ij} = 0$, *егер* $i < j$.

[1]-[2] жұмыстарында $1< p,q <\infty$ болғанда, $(a_{i,j})$ теріс емес матрицаның элементтері төмендегі 1-шарттын қанағаттандырғанда (2) операторы үшін (1) теңсіздіктің орындалуының қажетті және жеткілікті шарттары алынған:

1-шарты: $$d^{-1}(a_{ik} + a_{kj}) \le a_{ij} \le d(a_{ik} + a_{kj}), \quad i \ge k \ge j \ge 1$$

теңсіздіктері орындалатындай $d > 0$ тұрақтысы бар болсын.