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Control and inverse problems for the heat equation with strong singularities

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ABSTRACT

We consider a linear system composed of $N+1$ one dimensional heat equations connected by point-mass-like interface conditions. Assume an L^2 Dirichlet boundary control at one end, and Dirichlet boundary condition on the other end. Given any L^2-type initial temperature distribution, we show that the system is null controllable in arbitrarily small time. The proof uses known results for exact controllability for the associated wave equation. An argument using the Fourier Method reduces the control problem for both the heat equation and the wave equation to certain moment problems. Controllability is then proved by relating minimality properties of the family of exponential functions associated to the wave with the family associated to the heat equation. Based on the controllability result we solve the dynamical inverse problem, i.e. recover unknown parameters of the system from the Dirichlet-to-Neumann map given at a boundary point.

1. Introduction

There has been much interest in so called "hybrid systems" in which the dynamics of elastic systems and possibly rigid structures are related through some form of coupling. The study of controllability and stabilization of such structures has been made in a number of works, see [1] and references therein, The controllability of a string with a single attached mass was first considered in [2]. For the case of N attached masses, see in [3,4] and references therein. The controllability of a series of Euler–Bernoulli beams with interior attached masses was considered in [5], and for the Schrödinger equation with strong singularities see [6,7].

In this paper we consider the heat equation on the interval $[0,\ell]$ with strong singularities at $a_j,\ j=1,\dots,N$, where $0=a_0<a_1<\cdots<a_N<a_{N+1}=\ell$. In what follows, $v_j(x,t)$ will denote the temperature in the interval (a_j,a_{j+1}), and $h_j(t)$ will denote the temperature at $x=a_j$. We consider the following system. For $j=0,\dots,N$, we have

$$c_j(x)\rho_j(x)\frac{\partial v_j}{\partial t}-\left(\frac{\partial}{\partial x}k(x)\frac{\partial v_j}{\partial x}\right)+q_j(x)v_j=0,\ \ t>0,\ \ x\in(a_j,a_{j+1}), \tag{1.1}$$

while for $j=1,\dots,N$ we have

$$v_{j-1}(a_j^-,t)=h_j(t)=v_j(a_j^+,t),$$

$$k_j(a_j^+)\frac{\partial v_j}{\partial x}(a_j^+,t)-k_{j-1}(a_j^-)\frac{\partial v_{j-1}}{\partial x}(a_j^-,t)=\tilde{c}_j\tilde{M}_jh_j'(t), \tag{1.2}$$

$$v_N(\ell^-,t)=0. \tag{1.3}$$

Here $v(a_j^+,t):=\lim_{\epsilon\to0^+}v(a_j+\epsilon,t)$ for fixed t, and $v(a_j^-,t)$ is defined similarly. This system is an idealization of $(N+1)$ thin rods separated by point masses $\tilde{M}_j$ placed at a_j; c_j and $\tilde{c}_j$ are the specific heats of segments and masses, and ρ_j, resp. k_j, are the mass density, resp. thermal conductivity, of the segment (a_j,a_{j+1}). Finally, $q_j(x)$ represents some potential on (a_j,a_{j+1}).

We assume initial conditions

$$v_j(x,0)=w_j(x),\ \ x\in(a_j,a_{j+1}),\ j=0,\dots,N, \tag{1.4}$$

$$h_j(0)=(w)_j,\ j=1,\dots,N, \tag{1.5}$$

for $(2N+1)$-tuple $w:=(w_0(x),\dots,w_N(x),(w)_1,\dots,(w)_N)\in\oplus_{j=0}^NL^2(a_j,a_{j+1})\oplus\mathbb{R}^N$, and we assume a control is applied at $x=0$:

$$v_0(0^+,t)=f(t),\ \ t>0. \tag{1.6}$$

We assume for $j=0,1,2$, that q_j extends to $C[a_j,a_{j+1}]$, while for $j>2$, q_j extends to a function in $C^{j-2}[a_j,a_{j+1}]$. We assume $c_j,\ \rho_j,\ k_j\in C^{j+2}[a_j,a_{j+1}]$, and each of these functions is strictly

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positive, and also that $\tilde{M}_j$, $\tilde{c}_j$ are all positive. Our methods still apply if $q_j \in H^{\max(0,j-2)}(a_j, a_{j+1})$, but the presentation is more cumbersome.

Here and in what follows $H^j(a_j, a_{j+1})$ refers to the standard Sobolev space, with $H^0 = L^2$. In what follows, we will refer to the vector $(v_0, h_1, v_1, \ldots, h_N, v_N)$ simply as $v^f(x,t)$. Denote $L^2_M(0,\ell) = \oplus_{j=0}^N L^2(a_j, a_{j+1}) \oplus \mathbb{R}^N$. The definition of the norm $\| * \|_{L^2_M(0,\ell)}$ will be given in the next section.

The associated Sturm–Liouville problem has a self-adjoint operator with discrete spectrum, from which we define a family of Sobolev-like spaces $\mathcal{H}_p$. Using a standard Fourier series argument (see [8, Section III.1]) with estimates (2.10), (2.11), we have the following well-posedness result (analogous to the case without mass).

Theorem 1. *For any $f \in L^2_{loc}(0,\infty)$, there exists a unique solution $v^f(x,t)$ to the system (1.1)–(1.6). For any $T > 0$, the mapping $f \mapsto v^f(x,T)$ is a bounded map from $L^2(0,T)$ to $\mathcal{H}_{-1}$.*

Our first main result is the following

Theorem 2. *Let $w \in L^2_M$. For any $\tau > 0$, there exists $f \in L^2(0,\tau)$ such that $v^f(x,\tau) = 0$. Furthermore, there exists a constant C independent of w so that*

$$\|f\|_{L^2(0,\tau)} \le C\|w\|_{L^2_M(0,\ell)}. \tag{1.7}$$

In the case $N = 1$, System (1.1)–(1.6) was previously considered by Hansen and Martinez. The authors proved the well-posedness of the system in [9] and null controllability in [10]. Null controllability in the case $N = 1$ was also proved in [11]. The results of both papers [10] and [11] are based on analysis of spectral properties of the generalized Sturm–Liouville operator (operator A in Section 2). The methods in their papers cannot easily be extended to our setting with $N > 1$.

Our second main result concerns identifiability of System (1.1)–(1.6), where for simplicity for all j we set $q_j = 0$ and $k_j = c_j = \tilde{c}_j = 1$. We introduce the *response operator* R^T as the dynamical Dirichlet-to-Neumann map: $(R^T f)(t) = \frac{\partial v_0}{\partial x}(0^+, t)$, $0 < t < T$. We prove that for any $T > 0$, the response operator uniquely determines N, ℓ, and a_j, M_j, for $j = 1, 2, \ldots$, and ρ_j, for $j = 0, 1, 2, \ldots, N$. We also describe the algorithm reconstructing these parameters.

We conclude this section with a brief outline of the paper. In Section 2, we state some facts about the spectral theory of the associated Sturm–Liouville problem. Let λ_n, $n \in \mathbb{N}$, be the eigenfrequencies for this problem, with associated unit eigenfunctions φ_n. The proof of our theorem makes use of exact controllability results found in [3,4] for a related wave equation. The controllability problems for both the wave and the heat equation are reduced to moment problems. The exact controllability of the wave equation implies minimality for the exponential family $\{e^{i\lambda_n t}\}$, along with a certain estimate for the associated biorthogonal family. A result due to David Russell [12] then shows that $\{e^{-\lambda_n^2 t}\}$ is minimal on $L^2(0,T)$ for any $T > 0$, and gives an estimate for the biorthogonal set associated with *this* family. A standard Fourier series argument allows us to solve the moment problem for the heat equation. We also use an extension of the Russell result to show that the control functions can be chosen to be more regular than L^2; this will be used when solving the inverse problem. The discussion of the wave equation and the moment problems is found in Section 3. In Section 4 we solve our inverse problem using the ideas of the boundary control (BC) method. First, from R^T we reconstruct the so-called spectral data of System (1.1)–(1.6), the set $\{\lambda_n, \varphi_n'(0)\}$. Then we recover the parameters of the system using the results concerning the inverse problem for the corresponding string equation with attached masses [13]. To our best knowledge it is the first result concerning the inverse problem for the heat equation with strong singularities.

2. Sturm–Liouville problem and associated Sobolev spaces

We estimate some simplifying notation. Let $M_j = \tilde{c}_j \tilde{M}_j$, and define $\rho(x)$ by $\rho(x) = c_j(x)\rho_j(x)$ for $x \in (a_j, a_{j+1})$, and similarly define $q(x)$, $k(x)$. Then the associated Sturm–Liouville problem is:

$$-(k\phi')' + q\phi = \rho\lambda^2\phi, \; x \in (0,\ell) \setminus \{a_j\}_1^N, \tag{2.8}$$
$$\phi(\ell) = \phi(0) = 0,$$
$$\phi(a_j^-) = \phi(a_j) = \phi(a_j^+),$$
$$k(a_j^+)\phi'(a_j^+) - k(a_j^-)\phi'(a_j^-) = -\lambda^2 M_j \phi(a_j), \; j = 1, \ldots, N. \tag{2.9}$$

Associated to this problem is the Hilbert space $L^2_M(0,\ell) := L^2(0,\ell) \oplus \mathbb{R}^N$, which is defined as the completion of smooth functions on $[0, l]$ in the norm

$$\|v\|_{L^2_M} = \Big[\sum_{j=0}^N \Big(\int_{a_j}^{a_{j+1}} |v(x)|^2 \; \rho(x)dx\Big) + \sum_{j=1}^N M_j |v(a_j)|^2\Big]^{1/2}.$$

Denote by $\langle \cdot, \cdot \rangle_{L^2_M}$ the associated inner product.

Let $\{\lambda_n^2\}_{n=1}^\infty$ be the set of eigenvalues of System (2.8)–(2.9), listed in increasing order. Define $\ell_{op,j}$ be the optical length of the interval (a_j, a_{j+1}):

$$\ell_{op,j} = \int_{a_j}^{a_{j+1}} \sqrt{\frac{\rho(x)}{k(x)}}\, dx,$$

so the optimal length of the string is $\ell_{op} := \ell_{op,0} + \cdots + \ell_{op,N}$. Taking (possibly complex) square roots, we then define the associated eigenfrequencies $\Lambda := \{\lambda_n : \; n = \pm 1, \pm 2, \ldots\}$. Define $\mathbb{K} := \mathbb{Z} \setminus \{0\}$.

In [4] the following statements are proven.

Theorem 3. *(A) Let Λ' be any subset of Λ obtained by deleting $2N$ elements. Then Λ' can be reparametrized as*

$$\Lambda' = \bigcup_{j=0}^N \{\lambda_m^{(j)}\}_{m \in \mathbb{K}},$$

where for each j,

$$|\lambda_m^{(j)} - \frac{\pi m}{\ell_{op,j}}| = O(|m|^{-1}). \tag{2.10}$$

(B) The eigenvalues of System (2.8)–(2.9) are simple.

An immediate consequence of this theorem is that the frequencies are not uniformly separated, which makes the methods used in [14] or [15] not easy to apply.

We now define quadratic form

$$Q(u,v) = \sum_{j=0}^N \int_{a_j}^{a_{j+1}} [k(x)u'(x)v'(x) + q(x)u(x)v(x)]\, dx,$$

with domain

$$\mathcal{Q} = \{u \in L^2_M(0,\ell) : \; u|_{(a_j,a_{j+1})} \in H^1(a_j, a_{j+1}),$$
$$u(a_j^-) = u(a_j) = u(a_j^+) \; \forall j, \; u(0) = u(\ell) = 0\}.$$

Using [16, Theorem VIII.15], one can associate with this semi-bounded, closed form the self-adjoint operator A in $L^2_M(0,l)$, with operator domain

$$D(A) = \{u \in \mathcal{Q} : \; Au \in L^2_M(0,\ell)\}.$$

Then for $u \in D(A)$,

$$Au(x) = \begin{cases} -(ku')'(x) + q(x)u(x), & x \neq a_j,\ j = 1,\dots,N, \\ \frac{1}{M_j}\left((ku')(a_j^-) - (ku')(a_j^+)\right), & x = a_j,\ j = 1,\dots,N. \end{cases}$$

Let $\{\varphi_n\}_{n=1}^{\infty}$ be the set of normalized eigenfunctions of A. By the simplicity of the spectrum and the self-adjointness of A we have that this set is orthonormal with respect to $\langle\cdot,\cdot\rangle_{L^2_M}$. It is easy to check that φ_n solves the system (2.8)–(2.9) with $\lambda = \lambda_n$. The following result was proven in [4]:

$$|\varphi_n'(0)| \leq Cn, \tag{2.11}$$

where C is independent of n. We remark in passing that unlike the standard Sturm Liouville problem with $M_j = 0$ for all j, it is not always the case that $|\varphi_n'(0)| \asymp n$; see [2–4,6,17] for details.

We use the spectral representation to create a scale of Sobolev-like spaces:

$$\mathcal{H}_p = \{u(x) = \sum_{n=1}^{\infty} a_n\phi_n(x)\ :\ \|u\|_p^2 = \sum_{n=1}^{\infty} |a_n|^2(\lambda_n^2 + E)^p < \infty\},\quad p \in \mathbb{R}.$$

Here the constant E is chosen so that $\lambda_n^2 + E > 0$ for all n, and the infinite sums are understood to be the completion of finite sums with respect to the norm $\|\cdot\|_p$. Thus $\mathcal{H}_0 = L^2_M(0,l)$ and $\mathcal{H}_2 = D(A)$. Associated to these spaces are various symmetric compatibility conditions at $x = a_j$. To give an example, any eigenfunction ϕ is in $\mathcal{H}_n$ for all n, so that for each j we have

$$\frac{1}{M_j}\left((k\phi')(a_j^-) - (k\phi')(a_j^+)\right) = \lambda^2\phi(a_j) = \phi''(a_j^-) + (q\phi)(a_j^-) = \phi''(a_j^+) + (q\phi)(a_j^+).$$

For more details on these compatibility conditions, the reader is referred to [4].

3. Controllability as a moment problem

3.1. Associated wave equation

In what follows, we will write $j \geq 0$ to mean $j = 0,\dots,N$, etc. Consider the following system, which models a vibrating string with attached masses.

$$\rho(x)\frac{\partial^2 u_j}{\partial t^2} - \left(\frac{\partial}{\partial x}k(x)\frac{\partial u_j}{\partial x}\right) + q(x)u_j = 0,\ t > 0,\ x \in (a_j, a_{j+1}),\ j \geq 0, \tag{3.12}$$

$$u_{j-1}(a_j^-, t) = h_j(t) = u_j(a_j^+, t),\ j \geq 1, \tag{3.13}$$

$$\begin{aligned} k_j(a_j^+)\frac{\partial u_j}{\partial x}(a_j^+, t) - k_{j-1}(a_j^-)\frac{\partial u_{j-1}}{\partial x}(a_j^-, t) &= M_j h_j''(t),\ M_j > 0,\ j \geq 1, \\ u_N(\ell^-, t) &= 0, \\ u_j(x,0) = \frac{\partial u_j}{\partial t}(x,0) &= 0,\ x \in (a_j, a_{j+1}),\ j \geq 0, \\ h_j(0) = h_j'(0) &= 0,\ j \geq 1, \\ u_0(0^+, t) &= f(t),\ t > 0. \end{aligned} \tag{3.14}$$

For $f \in L^2(0,T)$, the weak solution u can be realized as a function of time with values in a generalized function space using the Fourier method, see [8, Section III.2]. In this section, we recall properties of solutions of the system (3.12)–(3.14), proven in [4]. Denoting X' to be the dual of X, we define $\theta^{-1}(0,a_1) := \{u \in H^1(0,a_1)\ :\ u(0) = 0\}'$. One of the most important features of System (3.12)–(3.14) is that the attached masses will mollify transmitted waves, so the system is well posed in asymmetric spaces. This is reflected in the following

Proposition 1. *For any $T > 0$, let $f \in L^2(0,T)$. There exists a unique solution*

$$u^f := (u_0, h_1, u_1, h_2, \dots, h_N, u_N)$$

to System (3.12)–(3.14). We have

$$u_0 \in C([0,T], L^2(0,a_1)) \cap C^1([0,T], \theta^{-1}(0,a_1))$$

and for $j = 1,\dots,N$ we have

$$u_j \in C([0,T]; H^j(a_j, a_{j+1})) \cap C^1([0,T], H^{j-1}(a_j, a_{j+1})).$$

Furthermore, $h_j \in H^j(0,T)$ for each j.

Since $f \in L^2$, the vector $(u_0, h_1, \dots, h_N, u_N)$ is not a classical solution to the system.

We say the pair of functions $(y_0(x), y_1(x))$ is in the "reachable set at time T" if there exists $f \in L^2(0,T)$ such that $(u^f(x,T), u_t^f(x,T)) = (y_0(x), y_1(x))$. We wish to characterize the reachable sets. To this end, we define

$$\tilde{W}_0 = \oplus_{j=0}^N H^j(a_j, a_{j+1}) \oplus \mathbb{R}^N \text{ and}$$
$$\tilde{W}_{-1} = \theta^{-1}(0,a_1) \oplus \left(\oplus_{j=1}^N H^{j-1}(a_j, a_{j+1})\right) \oplus \mathbb{R}^{N-1},$$

where the terms in $\mathbb{R}^N$ will account for the position of the masses at a_j, $j = 1,\dots,N$, while the terms in $\mathbb{R}^{N-1}$ will account for the velocity of the masses at a_j, $j = 2,\dots,N$. Because $h_1' \in L^2$, we cannot discuss $h_1'(T)$ in our framework. For each j, the masses impose on $(u^f(x,T), u_t^f(x,T))$ a set of equations that must hold at $x = a_j$, provided u^f and u_t^f are sufficiently regular. One example of this is $u^f(a_j^-, T) = h_j(T) = u^f(a_j^+, T)$, which by (3.13) and Proposition 1 must hold for all $j \geq 2$. In addition, the boundary condition at $x = \ell$ imposes further conditions. The collection of such equations satisfied by $u^f(x,T)$ will be denoted $\mathcal{C}_*^0$, while the collection of equations for $u_t^f(x,T)$ will be denoted $\mathcal{C}_*^{-1}$. These spaces are carefully described in [4]. We now define a Hilbert space W_i, for integers $i = -1, 0$, by

$$W_i = \{\phi \in \tilde{W}_i :\ \phi \text{ satisfies } \mathcal{C}_*^i\}.$$

The following inclusions are valid (see [4]):

$$\mathcal{H}_N \subset W_0,\ \ \mathcal{H}_{N-1} \subset W_{-1}.$$

In [4] we proved the exact controllability of System (3.12)–(3.14) in asymmetric spaces:

Theorem 4. *Let $N \geq 1$, and let $T > 2\ell_{op}$. Then for any $(y_0, y_1) \in W_0 \times W_{-1}$, there exists a control $f \in L^2(0,T)$ such that the solution u^f to (3.12)–(3.14) satisfies*

$$u^f(\cdot,T) = y_0,\ u_t^f(\cdot,T) = y_1.$$

Furthermore,

$$\|f\|^2_{L^2(0,T)} \asymp \|y_0\|^2_{W_0} + \|y_1\|^2_{W_{-1}}.$$

This result is sharp in the sense that the space of reachable functions $W_0 \times W_{-1}$ is the largest possible.

Corollary 1. *For any $T > 2\ell_{op}$, the system (3.12)–(3.14) is exactly controllable with respect to the symmetric space $\mathcal{H}_N \times \mathcal{H}_{N-1}$.*

Corollary 1 implies certain important properties for the exponential family of an associated moment problem, which we discuss now. For the rest of this section, we will assume for simplicity that the eigenvalues satisfy $\lambda_n^2 > 0$. If this were not the case in what follows, it would suffice to replace $\sin(\lambda t)/\lambda$ by $\sinh(|\lambda|t)/|\lambda|$ in the case $\lambda^2 < 0$, and by t in the case $\lambda^2 = 0$.

We present the solution of System (3.12)–(3.14) in the form of the series

$$u^f(x,t)=\sum_{n=1}^{\infty} a_n(t)\,\varphi_n(x). \tag{3.15}$$

For any $T>0$ and $f\in L^2(0,T)$, standard calculations using the weak solution formulation (see, e.g. [8, Ch. 3]) give for each n,

$$\begin{aligned} a_n(t)&=\frac{\varphi_n'(0)}{\lambda_n}\int_0^t f(\tau)\,\sin\lambda_n(t-\tau)\,d\tau,\\ a_n'(t)&=\varphi_n'(0)\int_0^t f(\tau)\,\cos\lambda_n(t-\tau)\,d\tau. \end{aligned} \tag{3.16}$$

We set $\alpha_n := \lambda_n a_n(T)$, $\beta_n := a_n'(T)$, and let $\langle\cdot,\cdot\rangle_T$ be the standard complex inner product on $L^2(0,T)$. Let $f^T(t)=f(T-t)$, so $\int_0^T f(t)\sin(\lambda(T-t))dt=\langle f^T,\sin(\lambda t)\rangle_T$. Using $e^{it}=\cos(t)+i\sin(t)$, we can rewrite (3.16) as

$$\begin{aligned} i\alpha_n+\beta_n &= \langle f^T, e^{-i\lambda_n t}\varphi_n'(0)\rangle_T,\\ -i\alpha_n+\beta_n &= \langle f^T, e^{i\lambda_n t}\varphi_n'(0)\rangle_T,\ \ \forall n\in\mathbb{N}. \end{aligned}$$

We define $\lambda_{-n}=-\lambda_n$ for $n\in\mathbb{N}$. Similarly we set $\alpha_{-n}=-\alpha_n$, and $\beta_{-n}=\beta_n$ for all $n\in\mathbb{N}$. Define γ_n by

$$\gamma_n=(-i\alpha_n+\beta_n),\ \ \forall\, n\in\mathbb{K}. \tag{3.17}$$

Then we have

$$\langle f^T, \varphi_n'(0)e^{i\lambda_n t}\rangle_T=\gamma_n,\ \ n\in\mathbb{K}. \tag{3.18}$$

Assigning terminal data to System (1.1)–(1.6) is equivalent to assigning values to $\{\gamma_n\}$, in which case (3.18) can be viewed as a moment problem. We will be considering terminal data in $\mathcal{H}_N\times\mathcal{H}_{N-1}$, and hence we can write $\gamma_n=\delta_n/(\lambda_n^2+E)^{(N-1)/2}$ with $\{\delta_n\}\in\ell^2$. Thus we can rewrite the moment problem as

$$\langle f^T, (\lambda_n^2+E)^{(N-1)/2}\varphi_n'(0)e^{i\lambda_n t}\rangle_T=\delta_n,\ \ n\in\mathbb{K}. \tag{3.19}$$

with $\{\delta_n\}$ ranging over ℓ^2. Since we have exact controllability with respect to $\mathcal{H}_N\times\mathcal{H}_{N-1}$, we can apply [8, Theorem III.3.10.b]; in the notation of that theorem, $r=-N+1$, $\rho_n=(\lambda_n^2+\alpha)^{r/2}$, and B is defined by

$$\langle Bf,\phi\rangle_{\mathcal{H}_r-\mathcal{H}_{-r}}=\int_0^T f(s)\overline{\phi'(0,s)}ds.$$

We conclude the family

$$\mathcal{E}=\{(\lambda_n^2+E)^{(N-1)/2}\varphi_n'(0)e^{i\lambda_n t}\,:\,n\in\mathbb{K}\}$$

is minimal in $L^2(0,T)$ for any $T>2\ell_{op}$, and the norms of the elements of the biorthogonal family $\{\tilde\xi_n\}$ associated to $\mathcal{E}$ are uniformly bounded: $\|\tilde\xi_n\|_{L^2(0,T)}\le C$, with C independent of n.

Proposition 2. *For any $T>2\ell_{op}$, the family $\{\varphi_n'(0)e^{i\lambda_n t}\,:\,n\in\mathbb{K}\}$ is minimal in $L^2(0,T)$, and has a biorthogonal family $\{\xi_n\}$ with*

$$\|\xi_n\|_{L^2(0,T)}\le C|n|^{N-1}. \tag{3.20}$$

Proof. Let $\xi_n=\tilde\xi_n(\lambda_n^2+E)^{(N-1)/2}$. Then this set is biorthogonal to $\{\varphi_n'(0)e^{i\lambda_n t}\}$, and the estimate (3.20) follows from (2.10).

3.2. Proof of Theorem 2

We recall the associated exponential sets. List the eigenvalues $\{\lambda_n^2\,:\,n\in\mathbb{N}\}$ in increasing order, and then we set $\lambda_{-n}=-\lambda_n$. In the case where zero is not an eigenvalue, we set

$$\mathcal{E}_h=\{\varphi_n'(0)e^{i\lambda_n t}\,:\,n\in\mathbb{K}\},$$

while

$$\mathcal{E}_p=\{\varphi_n'(0)e^{-\lambda_n^2 t}\,:\,n\in\mathbb{N}\}.$$

If $\lambda_m^2=0$ for some m, then zero is a double frequency, and

$$\mathcal{E}_h=\{\varphi_n'(0)e^{i\lambda_n t}\,:\,n\in\mathbb{K}\setminus\{m,-m\}\}\cup\{\varphi_n'(0),\varphi_n'(0)\,t\},$$

while

$$\mathcal{E}_p=\{\varphi_n'(0)e^{-\lambda_n^2 t}\,:\,n\in\mathbb{N}\setminus\{m\}\}\cup\{\varphi_n'(0)\}.$$

To prove Theorem 2, we will use the following result.

Proposition 3. *Let $T>0$. Suppose the family $\{\varphi_n'(0)e^{i\lambda_n t}\,:\,n\in\mathbb{K}\}$ is minimal in $L^2(0,T)$ with a biorthogonal family $\{\xi_n\}$. Then for any $\tau>0$, the family*

$$\{\varphi_n'(0)e^{-\lambda_n^2 t}\,:\,n\in\mathbb{N}\}$$

is minimal in $L^2(0,\tau)$, and one can choose the associated biorthogonal family $\{\theta_n\,:\,n\in\mathbb{N}\}$ to satisfy the estimates

$$\|\theta_n\|_{L^2(0,\tau)}\le C\|\xi_n\|_{L^2(0,T)}\,e^{\beta|\lambda_n|} \tag{3.21}$$

with constants C,β independent of n.

This result is a slight generalization of a result found in [8, Theorem II.5.20], which in turn is a generalization of the result due to Russell, [12]. The proof will be provided in the Appendix.

We now prove Theorem 2. We present the solution of System (1.1)–(1.6) in the form of the series

$$v^f(x,t)=\sum_{n=1}^{\infty} b_n(t)\,\varphi_n(x) \tag{3.22}$$

and the initial data from (1.5), (1.6) in the form: $w=\sum_{n=1}^{\infty} b_n^0\varphi_n$ with $\{b_n^0\}\in\ell^2$. For any $T>0$ and $f\in L^2(0,T)$, standard calculations (see, e.g. [8, Ch. III]) demonstrate that, for each $n\in\mathbb{N}$,

$$b_n(T)=\varphi_n'(0)\int_0^T f(t)\,e^{-\lambda_n^2(T-t)}\,dt\,+e^{-\lambda_n^2 T}b_n^0\,. \tag{3.23}$$

These equalities can be written as a moment problem

$$\langle f^T, \varphi_n'(0)e^{-\lambda_n^2 t}\rangle_T=b_n(T)-e^{-\lambda_n^2 T}b_n^0,\ \ n\in\mathbb{N}. \tag{3.24}$$

We claim System (1.1)–(1.6) is spectrally controllable in time τ with respect to $L^2_M(0,\ell)$, meaning that for zero initial data in (1.4), (1.5), and any $n\in\mathbb{N}$, there exists $f_n\in L^2(0,\tau)$ such that

$$v^{f_n}(x,\tau)=\varphi_n.$$

In fact, it is easy to see that $f_n=\theta_n$, $\forall n$.

The null controllability problem is equivalent to solvability of (3.24) with $T=\tau$ and $b_n(\tau)=0$, and clearly this moment problem is solved by the control

$$f(t)=-\sum_{n=1}^{\infty} e^{-\lambda_n^2\tau}b_n^0\,\theta_n(t).$$

This series converges and, by (3.21), we have

$$\|f\|_{L^2(0,\tau)}\le C\|w\|_{L^2_M},$$

completing the proof of Theorem 1. This proves null controllability of System (1.1)–(1.6) in the time interval $(0,\tau)$ with any $\tau>0$. $\square$

We now prove controllability with more regular control functions. This will be necessary in the next section in solving the inverse problem for our system. The following result is presented as a remark in [15], and its proof is sketched there. Since this result is essential for us, we give a more detailed proof here.

Proposition 4. *System (1.1)–(1.6) is spectrally and null controllable for any $\tau > 0$ with control space $H_0^2(0,\tau)$.*

Proof. Fix $\tau > 0$. We assume the eigenvalues are never zero for the moment, giving the modifications necessary in the other case at the end. We will prove null controllability, leaving the adaptations necessary for spectral control to the reader. Adapting the proof of Theorem 2, we will solve the moment problem

$$\langle f^\tau, \varphi_n'(0)e^{-\lambda_n^2 t}\rangle_\tau = -e^{-\lambda_n^2\tau} b_n^0,\ \ n \in \mathbb{N} \tag{3.25}$$

with $f \in H_0^2(0,\tau)$.

Let Λ_0 be the set of eigenvalues $\{\lambda_n^2\}$ of System (2.8)–(2.9) excluding zero if one of λ_n equals zero, and E be the closure in $L^2(0,\tau)$ of finite linear combinations of exponentials $\varphi_n'(0)\,e^{-t\lambda_n^2}$, $\lambda_n^2 \in \Lambda_0$. The family $\mathcal{E} := \{\varphi_n'(0)\,e^{-t\lambda_n^2},\ n : \lambda_n \in \Lambda_0\}$ is minimal. Let $\{\theta_n,\ n : \lambda_n^2 \in \Lambda_0\}$, be the family biorthogonal to $\mathcal{E}$ constructed in Proposition 3. It is a well known, see e.g. [15, p. 279], [18] that E does not contain any polynomial in t. Then simple linear algebra, which is left to the reader, allows to construct the functions $\theta_0, \theta_{-1}, \theta_{-2}$ in the orthogonal complement to E, which are biorthogonal to $1, t, t^2$:

$$\langle \theta_0, 1\rangle_\tau = \langle \theta_{-1}, t\rangle_\tau = \langle \theta_{-2}, t^2\rangle_\tau = 1,$$

and all other corresponding scalar products are zero.

In what follows, we set $c_n = -\lambda_n^4 e^{-\lambda_n^2\tau} b_n^0$ if $\lambda_n \neq 0$ and $c_n = -2b_n^0$ if $\lambda_n = 0$.

Case 1 If $\lambda_n \neq 0$ $\forall n$, we consider now the moment problem for a function $h \in L^2(0,\tau)$:

$$\langle h, \varphi_n'(0)e^{-\lambda_n^2 t}\rangle_\tau = c_n,\ \ n \in \mathbb{N}, \tag{3.26}$$

supplemented with the conditions

$$\langle h, 1\rangle_\tau = \langle h, t\rangle_\tau = 0. \tag{3.27}$$

One can check that a solution to (3.26), (3.27) is given in the form

$$h(t) = \sum_{n=1}^{\infty} c_n\theta_n(t) + c_0\theta_0(t) + c_{-1}\theta_{-1}(t), \tag{3.28}$$

where the coefficients c_0 and c_{-1} are determined by the equations

$$\sum_{n=1}^{\infty} c_n\langle\theta_n, 1\rangle_\tau + c_0 = 0, \quad \sum_{n=1}^{\infty} c_n\langle\theta_n, t\rangle_\tau + c_{-1} = 0. \tag{3.29}$$

Since $b_n^0 \in \ell^2$, it follows from (3.21), (2.10), and the definition of c_n, that the sum (3.28) converges in $L^2(0,\tau)$, and the series in (3.29) converge absolutely.

Case 2 If $\lambda_m = 0$ for some $m \in \mathbb{N}$, we consider the following moment problem for a function h :

$$\langle h, \varphi_n'(0)e^{-\lambda_n^2 t}\rangle_\tau = c_n,\ \ n \in \mathbb{N},\ n \neq m; \quad \langle h, t^2\rangle_\tau = c_m, \tag{3.30}$$

supplemented with the conditions (3.27). The solution of (3.30), (3.27) is presented by

$$h(t) = \sum_{n\neq m}^{\infty} c_n\theta_n(t) + c_0\theta_0(t) + c_{-1}\theta_{-1}(t) + c_{-2}\theta_{-2}(t),$$

where the coefficients c_0, c_{-1} and c_{-2} are determined by equations

$$\sum_{n\neq m}^{\infty} c_n\langle\theta_n, 1\rangle_\tau + c_0 = 0, \quad \sum_{n\neq m}^{\infty} c_n\langle\theta_n, t\rangle_\tau + c_{-1} = 0,$$

$$\sum_{n\neq m}^{\infty} c_n\langle\theta_n, t^2\rangle_\tau + c_{-2} = 2c_m.$$

In both cases, one can then easily verify that the function

$$f^\tau(t) = \int_0^t (t-s)h(s)\,ds \tag{3.31}$$

solves (3.25), and evidently $f \in H_0^2(0,T)$. □

Remark 1. It is possible to strengthen Proposition 4 to proof null controllability with $H_0^n(0,\tau)$ controls for any $n \in \mathbb{N}$ by adapting the arguments of this paper.

Remark 2. An analogous result is proven [4], also see [19], for the vibrating strings with attached masses. To state this result more precisely, let us first we assume that the λ_n are all non-zero. Recall we have chosen the constant E so that the operator $A+E$ is strictly positive. Define $W_j = (A+E)^{-1}W_{j-2}$, and let $p \in \mathbb{N}$. In [4], Theorem 7.6, it is stated that for $T > 2\ell_{op}$, the system (1.1)–(1.6) with state space $W_p \times W_{p-1}$ is exactly controllable with controls in $H_0^p(0,T)$, but there is a gap in the proof there that we will correct with this paragraph. It was proven in ([20], Theorem 3) that for $T > 2\ell_{op}$, the family of generalized exponential divided differences composed of

$$\{e^{i\lambda_n t}\ :\ n \in \mathbb{K}\} \cup \left(\bigcup_{j=0}^{p}\{t^j\}\right)$$

forms a Riesz sequence in $L^2(0,T)$. As a consequence, in the proof of Theorem 7.6 [4], the function f (in the notation of that paper) can be chosen so that $\langle f^T, t^j\rangle_T = 0, j = 0,\dots,p-1$, and this validates the identity $\frac{d^p g}{dt^p} = f$ in that paper. The rest of the argument proceeds as in that paper. Thus Theorem 7.6 is proven only for $T > 2\ell_{op}$, and not sometimes for $T = 2\ell_{op}$ as stated there.

4. Inverse problem

Let $T > 0$. We consider the following special case of (1.1)–(1.6)

$$\rho_j(x)\frac{\partial v_j}{\partial t} - \frac{\partial^2 v_j}{\partial x^2} = 0,\ \ t > 0,\ \ x \in (a_j, a_{j+1}),\ \ j = 0,\dots,N, \tag{4.32}$$

$$\begin{aligned} v_{j-1}(a_j^-, t) &= h_j(t) = v_j(a_j^+, t),\ \ j = 1,\dots,N,\\ M_j h_j'(t) &= \frac{\partial v_j}{\partial x}(a_j^+, t) - \frac{\partial v_{j-1}}{\partial x}(a_j^-, t),\ \ j = 1,\dots,N,\\ v_N(\ell^-, t) &= 0,\\ v_j(x,0) &= 0,\ \ \ x \in (a_j, a_{j+1}),\ \ j = 0,\dots,N,\\ h_j(0) &= 0,\ \ j = 1,\dots,N,\\ v_0(0^+, t) &= f(t),\ \ t > 0. \end{aligned} \tag{4.33}$$

In what follows, we will drop the j subscript from ρ_j, v_j without any confusion.

To state our inverse problem, define the response operator, R^T, by

$$(R^T f)(t) = v_x^f(0,t),\ \ t \in (0,T).$$

An exercise in Fourier series, but also see [21, Eq. 9.17 on p.198], shows that R^T, defined classically on smooth functions vanishing to infinite order at $t = 0$, extends to a continuous mapping $L^2(0,T) \mapsto H^{-1}(0,T)$; here $H^{-1}(0,T)$ is the dual of $H_0^1(0,T)$. Our dynamical inverse problem is to recover $l, N, \{a_j\}, \{M_j\}$, and ρ, from R^T.

A key step in the procedure below is to recover the spectral data $\{\lambda_n^2, \varphi_n'(0)\}$ associated to the system. The spectral data is obtained by a variational argument, but to justify this argument we must first prove some regularity results.

From (3.23), (2.10) and (2.11) it follows that $\{b_n(T)\} \in \ell^2$ if $f \in H_0^1(0,T)$, and so $v^f(\cdot,T) \in L^2_M$. Therefore, the *control operator*

$$U^T : L^2(0,T) \mapsto L^2_M, \;\; U^T f = v^f(\cdot,T)$$

is well defined for $f \in H_0^1(0,T)$. We define the *connecting operator* C^T in $L^2(0,T)$ as $C^T = (U^T)^* U^T$ or, equivalently, by its bilinear form

$$(C^T f, g)_{L^2(0,T)} = \langle v^f(\cdot,T), v^g(\cdot,T)\rangle_{L^2_M}, \;\; f,g \in H_0^1(0,T).$$

By (3.22) and (3.23) with $b_n^0 = 0$, we have

$$v^f(x,t) = \sum_{n=1}^{\infty} \varphi_n(x)\varphi_n'(0)\int_0^t f(s)e^{-\lambda_n^2(t-s)}ds.$$

If $f \in H_0^1(0,T)$, then for all n with $\lambda_n \neq 0$ we have

$$\int_0^t f(s)e^{-\lambda_n^2(t-s)}ds = \frac{1}{\lambda_n^2}\left(-\int_0^t f'(s)e^{-\lambda_n^2(t-s)}ds + f(t)\right). \tag{4.34}$$

Thus, $t \mapsto v^f(x,t)$ is a continuous mapping $[0,T]$ to $L^2_M(0,l)$, and

$$\langle C^T f, g\rangle_{H^{-1}(0,T)-H_0^1(0,T)} = \langle v^f(\cdot,T), v^g(\cdot,T)\rangle_{L^2_M}, \;\; \forall f,g \in H_0^1(0,T). \tag{4.35}$$

Our test functions for the variational argument will be in $H_0^2(0,T)$. Supposing $f \in H_0^2(0,T)$, then by using (4.34), we have

$$v_t^f(x,t) = \sum_{n=1}^{\infty} \varphi_n(x)\frac{\varphi_n'(0)}{\lambda_n^2}\Big(-\int_0^t f''(s)e^{-\lambda_n^2(t-s)}ds + f'(t)\Big),$$

and the estimate

$$\Big|\int_0^t f'(s)e^{-\lambda_n^2(t-s)}ds\Big| \le C\frac{\|f'\|_{L^2(0,T)}}{n}$$

shows that $t \mapsto v_t^f(\cdot,t) = v^{f'}(\cdot,t)$ is a continuous map from $[0,T]$ to L^2_M. Hence

$$\langle C^T f', g\rangle_{H^{-1}(0,T)-H_0^1(0,T)} = \langle v_t^f(\cdot,T), v^g(\cdot,T)\rangle_{L^2_M}, \;\; \forall f,g \in H_0^2(0,T). \tag{4.36}$$

The following result is well known for the parabolic equations with regular coefficients [14,22].

Proposition 5. *The operator C^T can be explicitly expressed through the response operator:*

$$C^T = (Z^T)^* Y^{2T} R^{2T} Z^T, \tag{4.37}$$

where $Z^T : L^2(0,T) \mapsto L^2(0,2T)$ is given by the extension operator $(Z^T f)(t) = f(t)$ for $t \in [0,T]$ and zero otherwise. Its adjoint, acting from $L^2(0,2T)$ to $L^2(0,T)$, coincides with the restriction operator $(Z^T)^* f = f|_{[0,T]}$. Also $(Y^{2T} f)(t) = f(2T-t), \; t \in [0,2T]$.

Proof. We define the function $\zeta(s,t) = \langle v^f(\cdot,s), v^g(\cdot,t)\rangle_M$ for $f,g \in H_0^1(0,T)$, and extend f by zero for $t \in [T,2T]$. Clearly, $(C^T f,g)_{L^2(0,T)}$ does not depend on the extension of f. We will use the fact that $\zeta(T,T) = (C^T f,g)_{L^2(0,T)}$. Because $\rho v_t = v_{xx}$,

$$\begin{aligned}\zeta_t(s,t) - \zeta_s(s,t) &= \int_0^\ell [v^f(x,s)v_t^g(x,t) - v_s^f(x,s)v^g(x,t)]\rho(x)\,dx \\ &\quad + \sum_j M_j[v^f(a_j,s)v_t^g(a_j,t) - v_s^f(a_j,s)v^g(a_j,t)], \\ &= [v^f(x,s)v_x^g(x,t) - v_x^f(x,s)v^g(x,t)]_{x=0}^{\ell}, \\ &= (R^T f)(s)g(t) - f(s)(R^T g)(t).\end{aligned}$$

The following boundary conditions hold for ζ: $\zeta(s,0) = \zeta(0,t) = 0$, so solving for ζ by the method of characteristics, we get

$$\zeta(s,t) = \int_s^{s+t} [(R^T f)(\xi)g(s+t-\xi) - f(\xi)(R^T g)(s+t-\xi)]\,d\xi, \quad s,t \le T.$$

Therefore,

$$\begin{aligned}\zeta(T,T) &= \int_T^{2T} [(R^T f)(\xi)g(2T-\xi) - f(\xi)(R^T g)(2T-\xi)]\,d\xi \\ &= \int_0^T [(Rf)(2T-\xi)g(\xi) - f(2T-\xi)(Rg)(\xi)]\,d\xi \\ &= \int_0^T (Rf)(2T-\xi)g(\xi)\,d\xi.\end{aligned}$$

From this, (4.37) easily follows.

The operator C^T serves as a model for the operator A in the space of controls $L^2(0,T)$. Based on the mini-max principle and using the spectral controllability of our system (Proposition 4), one can recover the spectral data $\{\lambda_n, \varphi_n'(0)\}, n \in \mathbb{N}$ of operator A using the connecting operator C^T. This important result of the boundary control method is described in a series of papers (see, e.g. [14,22,23]). We demonstrate how to adjust this technique to our situation, and show we can recover the spectral data from R^T. Suppose $f_n \in H_0^2(0,T)$ satisfies $U^T f_n = \varphi_n$. It is well known that $\{\lambda_n^2, \varphi_n\}$ can be recovered from A using the mini-max principle. In our case, A is not known, but below we show how C^T can be used to find $\{\lambda_n^2, f_n\}$, adapting an argument found in [14,22,23]. Since $\varphi_n \in \mathcal{H}_1$, we have

$$\begin{aligned}\langle C^T f_n', f_n\rangle_{H^{-1}(0,T)-H_0^1(0,T)} &= \langle v^{f_n'}(\cdot,T), \varphi_n\rangle_{L^2_M} \\ &= \langle v_t^{f_n}(\cdot,T), \varphi_n\rangle_{L^2_M} \\ &= \langle A v^{f_n}(\cdot,T), \varphi_n\rangle_{\mathcal{H}_{-1}-\mathcal{H}_1} \\ &= \lambda_n^2.\end{aligned}$$

This, along with (4.35), allows one to use a variational argument parallel to the minimax argument for A to compute the set $\{\lambda_n^2, f_n\}$. Finally, by the definition of R^T, we have

$$R^T f_n(T) = v_x^{f_n}(0,T) = \varphi_n'(0).$$

Knowing the spectral data, we can construct the connecting operator C_w^T for the wave equation with masses, System (3.12)–(3.14) with $k=1, \;\; q=0$. Specifically, by (3.15), (3.16), we have for $f,g \in L^2(0,T)$,

$$(C_w^T f, g)_{L^2(0,T)} := \langle u^f(\cdot,T), u^g(\cdot,T)\rangle_{L^2_M} = \sum_{n=1}^{\infty} a_n^f(T)\,a_n^g(T)$$

$$= \sum_{n=1}^{\infty}\left|\frac{\varphi_n'(0)}{\lambda_n}\right|^2 \int_0^T f(t)\,\sin\lambda_n(T-t)\,dt \int_0^T g(t)\,\sin\lambda_n(T-t)\,dt.$$

By (2.10), (2.11), the series is absolutely convergent and bounded above by a constant times $\|f\|_{L^2(0,T)}\,\|g\|_{L^2(0,T)}$.

From the operator C_w^T one can recover ℓ, N, a_j, M_j, for $j = 1,\dots,N$, and $\rho(x)$. The algorithm is described in [13, Section 4].

Conclusions

We have proved null controllability and solved an inverse problem for the heat equation with the presence of a finite number of strong singularities. Our approach combines dynamical and spectral methods (i.e. frequency and time domain methods) and uses connections between controllability/identifiability of the wave and heat equations. This approach will provide much more simple solutions to control and inverse problems than solutions based on purely spectral methods, which are complicated even in the case of one singularity.

CRediT authorship contribution statement

Sergei Avdonin: Conceptualization, Investigation, Methodology, Writing, Editing. **Nina Avdonina:** Conceptualization, Investigation, Methodology. **Julian Edward:** Conceptualization, Investigation, Methodology, Writing, Editing. **Karlygash Nurtazina:** Conceptualization, Investigation, Methodology.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

Here we present a more general version of Proposition 3, covering vector valued exponentials. This version is a slightly modified version of [8, Theorem II.5.20]. The modifications serve to handle the special cases where there exist non-positive eigenvalues, and also to clarify the exposition and correct some misprints.

Recall we list the eigenvalues $\{\lambda_n^2 : n \in \mathbb{N}\}$ in increasing order, and then we set $\lambda_{-n} = -\lambda_n$. Given a Hilbert space V, of a finite or infinite dimension, let $\eta_n \in V$ for all $n \in \mathbb{N}$, and set $\eta_{-n} = \eta_n$. In the case where zero is not an eigenvalue, we set

$$\mathcal{E}_h = \{\eta_n e^{i\lambda_n t} : n \in \mathbb{K}\},$$

while

$$\mathcal{E}_p = \{\eta_n e^{-\lambda_n^2 t} : n \in \mathbb{N}\}.$$

If $\lambda_j^2 = 0$ for some $j \in \mathbb{N}$, then zero is a double frequency, and

$$\mathcal{E}_h = \{\eta_n e^{i\lambda_n t} : n \in \mathbb{K} \setminus \{j, -j\}\} \cup \{e_j, e_{-j}\}, \quad e_j := \eta_j,\ e_{-j} := \eta_j t,$$

while

$$\mathcal{E}_p = \{\eta_n e^{-\lambda_n^2 t} : n \in \mathbb{N} \setminus \{j\}\} \cup \{\eta_j\}.$$

Of course, for the purpose of this paper, it suffices to set $\eta_n = \varphi_n'(0)$.

Proposition 6. *Let $T > 0$. Suppose the family $\mathcal{E}_h$ is minimal in $L^2(0, T; V)$ with a biorthogonal family $\{\xi_n\}$. Then for any $\tau > 0$, the family $\mathcal{E}_p$ is minimal in $L^2(0, \tau; V)$, and one can choose the associated biorthogonal family $\{\theta_n : n \in \mathbb{N}\}$ to satisfy the estimates*

$$\|\theta_n\|_{L^2(0,\tau;V)} \le C\|\xi_n\|_{L^2(0,T;V)} e^{\beta|\lambda_n|} \tag{A.38}$$

with constants C, β independent of n.

Proof of Proposition 6. The main ingredients of the proof are a Paley–Wiener Theorem-type argument and complex interpolation, used to construct the biorthogonal set satisfying (A.38).

We denote the complex inner product on V by $v_0 \cdot v_1$ and set

$$\langle u_0, u_1\rangle_{L^2(0,T;V)} := \int_0^T u_0(x) \cdot u_1(x)\, dx, \quad u_0, u_1 \in L^2(0, T; V).$$

We will first prove the proposition when all eigenvalues λ_n^2 are nonnegative. The modifications necessary in the other case will be stated in the end. In either case, in what follows we denote $\mathcal{E}_h = \{e_n\}$ and $\mathcal{E}_p = \{\tilde{e}_n\}$.

Step 1:

Introduce for $m \in \mathbb{N}$ entire functions

$$\hat{G}_m(z) := \int_0^T e^{-izt}\xi_m(t)dt,\ z \in \mathbb{C},$$

and set $\tilde{G}_m(z) = \hat{G}_m(z) + \hat{G}_m(-z)$, so $\tilde{G}_m$ is double the even part of $\hat{G}_m$. Clearly,

$$\|\tilde{G}_m(z)\|_V \le \sqrt{T}\alpha_m e^{|z|T},\ \alpha_m = \|\xi_m\|_{L^2(0,T;V)}. \tag{A.39}$$

We claim

$$\tilde{G}_m(\lambda_n) \cdot \eta_n = \delta_{mn},\ m, n \in \mathbb{N}, \tag{A.40}$$

where δ_{mn} is the Kronecker delta function. The claim follows from the following two calculations:

$$\hat{G}_m(\lambda_n) \cdot \eta_n = \int_0^T e^{-i\lambda_n t}\xi_m(t) \cdot \eta_n\, dt = \langle \xi_m, e_n\rangle_{L^2(0,T;V)} = \delta_{mn},$$

$$\hat{G}_m(-\lambda_n) \cdot \eta_n = \int_0^T e^{i\lambda_n t}\xi_m(t) \cdot \eta_n\, dt = \langle \xi_m, e_{-n}\rangle_{L^2(0,T;V)} = 0.$$

Adding these two equations gives (A.40).

Step 2: We wish to relate the minimality of the exponential families associated to $\{\lambda_n\}$ and $\{i\lambda_n^2\}$. It is thus natural to pass from functions of z to functions of z^2. Since $\tilde{G}_m$ is an even function, it may be represented as a power series of even powers of z, so the function

$$Q_m(k) := \tilde{G}_m(\sqrt{k/i})$$

is well defined as an entire function. Then by (A.39), (A.40), there exists $C > 0$ so that we have:

$$\|Q_m(k)\|_V \le C\sqrt{T}\exp(T\sqrt{|k|})\alpha_m, \tag{A.41}$$

$$Q_m(i\lambda_n^2) \cdot \eta_n = \delta_{mn}. \tag{A.42}$$

Step 3:

The functions $\|Q_m\|_V$ generally may be increasing on horizontal lines. To make the upcoming Fourier transform argument possible, we multiply Q_m by a function $E(z)$ whose existence and properties are described in the following statement found in Fattorini and Russell [15]. In what follows, we write $z = x + iy$ with $x, y \in \mathbb{R}$.

Proposition 7. *For every $\tau > 0$, there exists a function $E(z)$ of exponential type such that its zeros are real and differ from zero; E is real on the real axis, and there exist positive constants β and $c_j(\tau)$ such that*

$$(i)\ |E(x)| \le c_1(\tau)\exp(-T\sqrt{|x|})/(1 + |x|), \tag{A.43}$$

$$(ii)\ \text{for } y \le 0,\ |E(iy)| \le c_1(\tau)\exp(\tau|y|), \tag{A.44}$$

$$(iii)\ \text{for } y \ge 0,\ 1 \ge |E(iy)| \ge c_3(\tau)\exp(-\beta\sqrt{y}). \tag{A.45}$$

Fix $\tau > 0$, and the corresponding function E, and define $G_m(z) := E(z)Q_m(z)$, and then define

$$\theta_m(t) := \frac{1}{2\pi E(i\lambda_m^2)} \int_{\mathbb{R}} e^{-ist} G_m(s) ds. \tag{A.46}$$

It follows from (A.41) and (A.43) that this Fourier transform converges. The functions $\{\theta_m\}$ will give us our biorthogonal family.

Lemma 1. *The functions θ_m have the following properties:*
(A) The support of θ_m is in $[0, \tau]$,
(B) $\|\theta_m\|_{L^2(0,\tau;V)} \leq c(\tau)\alpha_m \exp(\beta|\lambda_m|)$.

Proof. The proof of part A is an adaptation of the proof of the Paley–Wiener Theorem. By (A.41) and Proposition 7, the function G_m is of exponential type. Choose any ϵ with $0 < \epsilon < \tau$. We claim there exists a constant $C > 0$, independent of m and y, such that for $y \geq 0$,

$$\|e^{i\epsilon(x+iy)} G_m(x+iy)(x+iy+i)\|_V < C.$$

This follows because (A.41), (A.43), and (A.45) allow us to apply the Phragmén–Lindelöf Theorem in the two quarter spaces in $\mathbb{C}_+$. It follows that

$$\int_{x\in\mathbb{R}} \|e^{i\epsilon(x+iy)} G_m(x+iy)\|_V^2 \, dx < C, \;\; \forall y \geq 0,$$

with C independent of y. Hence, for $t < -\epsilon$, we have by (A.46) and deformation of integration contours,

$$\begin{aligned} 2\pi E(i\lambda_m^2)\theta_m(t) &= \int_{\mathbb{R}} e^{-ix(t+\epsilon)} e^{ix\epsilon} G_m(x) \, dx \\ &= \int_{\mathbb{R}} e^{-i(x+iy)(t+\epsilon)} e^{i(x+iy)\epsilon} G_m(x+iy) \, dx. \end{aligned}$$

Letting $y \to \infty$, this last term vanishes, proving $\theta_m(t) = 0$ for $t < -\epsilon$. A similar argument shows that $\theta_m(t) = 0$ for $t > \tau + \epsilon$. Letting $\epsilon \to 0$, part A follows.

Using part A of this lemma, formulas (A.46), (A.41) and property (iii) of Proposition 7, we have

$$\begin{aligned} \|\theta_m\|_{L^2(0,\tau;V)} &= \|\theta_m\|_{L^2(\mathbb{R};V)} \\ &= \frac{\|G_m\|_{L^2(\mathbb{R}:V)}}{\sqrt{2\pi}|E(i\lambda_m^2)|} \\ &\leq \sqrt{T/2\pi} \exp(\beta|\lambda_m|) c_3(\tau)^{-1} c_1(\tau) \\ &\quad \times \alpha_m \left(\int_{\mathbb{R}} 1/(1+|s|)^2 ds \right)^{1/2}. \end{aligned}$$

This proves part B.

We now verify that $\{\theta_m : m \in \mathbb{N}\}$ is biorthogonal to $\{e^{-t\lambda_n^2}\eta_n : n \in \mathbb{N}\}$ in the space $L^2(0, \tau; V)$. We use the Fourier inverse formula, and then part A of the lemma, to get

$$G_m(s) = E(i\lambda_m^2) \int_{\mathbb{R}} \theta_m(t) e^{ist} \, dt = E(i\lambda_m^2) \int_0^\tau \theta_m(t) e^{ist} \, dt.$$

Hence by (A.42),

$$\begin{aligned} \langle \theta_m, \tilde{e}_n \rangle_{L^2(0,\tau;V)} &= \int_0^\tau \theta_m(t) \cdot e^{-\lambda_n^2 t} \eta_n \, dt \\ &= \frac{G_m(i\lambda_n^2) \cdot \eta_n}{E(i\lambda_m^2)} \\ &= Q_m(i\lambda_n^2) \cdot \eta_n \frac{E(i\lambda_n^2)}{E(i\lambda_m^2)} \\ &= \delta_{mn}. \end{aligned}$$

The estimate (A.38) was proven in the part B of the lemma. We have proven the proposition when all eigenvalues are positive.

We now indicate how to adapt the argument in case we have negative, but not zero, eigenvalues. If $\lambda_j^2 = -\gamma^2$, with $\gamma > 0$, so that $\lambda_j = i\gamma$ and $\lambda_{-j} = -i\gamma$, then the proof of (A.40) goes as follows:

$$\hat{G}_m(\lambda_j) \cdot \eta_j = \int_0^T e^{\gamma t} \xi_m(t) \cdot \eta_j \, dt = \langle \xi_m, e_{-j} \rangle_{L^2(0,T;V)} = 0,$$

$$\hat{G}_m(\lambda_{-j}) \cdot \eta_{-j} = \int_0^T e^{-\gamma t} \xi_m(t) \cdot \eta_j \, dt = \langle \xi_m, e_j \rangle_{L^2(0,T;V)} = \delta_{mj}.$$

Adding these two equations again gives (A.40). The argument above then carries over until Proposition 7, where we must arrange that $E(x)$ does not vanish on any of the real roots of Q_m. If E vanishes on the set $\lambda_{m_1}, \ldots, \lambda_{m_p}$, then we can replace $E(z)$ by $E(z)/(\prod_{k=1}^p (z - \lambda_{m_k}))$, and the estimates in Proposition 7 still hold with possibly other constants $c_j(\tau)$. The rest of the proof of the proposition carries through without change.

We now consider the case where one eigenvalue is zero, say $\lambda_j = 0$. Then we define $\hat{G}_m$ and $\tilde{G}_m$ for $m \neq j$ as in Step 1 and introduce

$$\hat{G}_j(z) := \int_0^T e^{-izt} \xi_j(t) dt, \quad \text{and set} \quad \tilde{G}_j(z) = \frac{1}{2}\left[\hat{G}_j(z) + \hat{G}_j(-z)\right].$$

One can check that

$$\tilde{G}_m(\lambda_j) \cdot \eta_j = \langle \xi_m, e_j \rangle_{L^2(0,T;V)} = 0, \quad j \neq m,$$

$$\tilde{G}_j(\lambda_m) \cdot \eta_m = \frac{1}{2}[\langle \xi_j, e_m \rangle_{L^2(0,T;V)} + \langle \xi_j, e_{-m} \rangle_{L^2(0,T;V)}] = 0, \quad j \neq m,$$

$$\tilde{G}_j(\lambda_j) \cdot \eta_j = \langle \xi_j, e_j \rangle_{L^2(0,T;V)} = 1.$$

The rest of the proof of the proposition carries through without change. $\square$

Our proof works also in the case when $\lambda_j^2 = 0$ for some $j \in \mathbb{N}$, and the family $\mathcal{E}_h$ does not contain the element $e_{-j}(t) = \eta_j t$.

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