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MODELING NONLINEAR DEPENDENCIES USING GENERALIZED MEAN

Bralina Sabina
sabinabralinasabina@gmail.com
Euarsian National University, Nur-Sultan, Kazakhstan
Scientific adviser – Iskakova A.

Nowadays, statistical modeling is an indisputable way of understanding processes of any kind that helps us to avoid adverse outcomes or even to forecast the future tendencies. The majority of these tasks are dedicated to recreation of function by given set of its values. Estimation of its parameters are conducted using the apparatus of the theory of probability and mathematical statistics and it can be concluded that linear dependences do not always adequately describe the processes under study. Linear dependences are considered only as a special case for the convenience and clarity of considering the process under study. More often, there are models that reflect processes in the form of nonlinear dependence.

For example: damped harmonic and non-harmonic oscillations, which can characterize the sales volumes of seasonal goods at the stage of leaving the market. Or dependencies that characterize the life cycle of a product, remarketing or product conversion.[1]

The most popular method for processing experimental data was developed for almost two hundred years ago by A.M. Legendre and C.F. Gauss and is called regression analysis, or the method of least squares (OLS). Method is based on the minimization of the square value of the deviation between the sought analytical function and the experimental data.[2] In the case of the nonlinear dependency the given method can be used only with one condition: the possibility of transformation of the given variables to a linear dependency. This property is possessed by equations that are linear with respect to the parameters.[3] Through this condition nonlinear dependency of the form $\psi(y) = a\varphi(x) + b$ can be transformed into:

$$M'_i(x'_i, y'_i) = M'_i(\varphi(x_i), \psi(y_i))$$

by placing given set of values into a new coordinate plane $x'o\,y'$.

The form of given dependency can diverse, from simple square to exponential functions, but these functions are united by a common property concerning the generalized mean. The generalized mean of positive numbers $z_1, z_2, \dots, z_n$ is called

$$M_z(t) = \left(\frac{1}{n}\sum_{k=1}^{n} z_k^t\right)^{1/t}$$

Theorem 1.[4] The value of these functions at the point corresponding to the generalized mean over the argument x is equal to the generalized mean over the variable y, i.e.

$$f(M_x(p)) = M_y(q)$$

The proof of the theorem will be given for 25 distinct and the most common functional dependencies, which are considered in the thesis.

Proof:

1. $y = ax + b$, by hypothesis the given function corresponds with arithmetic mean for both variables x and y:

$$x^* = \frac{x_1 + x_2}{2} \Rightarrow y(x^*) = a\left(\frac{x_1 + x_2}{2}\right) + b$$

$$y^* = \frac{y_1 + y_2}{2} \Rightarrow y^* = \frac{ax_1 + b + ax_2 + b}{2} \Rightarrow y^* = \frac{a(x_1 + x_2)}{2} + b$$

2. $y = ba^x$, by hypothesis the given function corresponds with arithmetic mean for x and geometric mean for y:

$$x^* = \frac{x_1 + x_2}{2} \Rightarrow y(x^*) = ba^{\frac{x_1 + x_2}{2}}$$

$$y^* = \sqrt{y_1 y_2} \Rightarrow y^* = \sqrt{ba^{x_1} ba^{x_2}} \Rightarrow y^* = b\sqrt{a^{x_1} a^{x_2}} \Rightarrow y^* = ba^{\frac{x_1 x_2}{2}}$$

3. $y = \frac{1}{ax + b}$, by hypothesis the given function corresponds with arithmetic mean for x and harmonic mean for y:

$$x^* = \frac{x_1 + x_2}{2} \Rightarrow y(x^*) = \frac{1}{a\left(\frac{x_1 + x_2}{2}\right) + b}$$

$$y^* = \frac{2}{\frac{1}{y_1} + \frac{1}{y_2}} \Rightarrow y^* = \frac{2}{ax_1 + b + ax_2 + b} \Rightarrow y^* = \frac{2}{a(x_1 + x_2) + 2b} \Rightarrow y^* = \frac{1}{a\frac{(x_1 + x_2)}{2} + b}$$

4. $y = a\ln(x) + b$, by hypothesis the given function corresponds with geometric mean for x and arithmetic mean for y:
5.

$$x^* = \sqrt{x_1 x_2} \Rightarrow y(x^*) = a\ln(\sqrt{x_1 x_2}) + b \Rightarrow y(x^*) = \frac{a}{2}\ln(x_1 x_2) + b$$

$$y^* = \frac{y_1 + y_2}{2} \Rightarrow y^* = \frac{a\ln(x_1) + b + a\ln(x_2) + b}{2} \Rightarrow y^* = \frac{a(\ln(x_1) + \ln(x_2))}{2} + b \Rightarrow y^* = \frac{a\ln(x_1 x_2)}{2} + b$$

6. $y = b + \frac{a}{x}$, by hypothesis the given function corresponds with harmonic mean for x and arithmetic mean for y:

$$x^* = \frac{2}{\frac{1}{x_1} + \frac{1}{x_2}} \Rightarrow y(x^*) = b + \frac{a}{\frac{2}{\frac{1}{x_1} + \frac{1}{x_2}}} \Rightarrow y(x^*) = b + \frac{a\left(\frac{1}{x_1} + \frac{1}{x_2}\right)}{2}$$

$$y^* = \frac{y_1 + y_2}{2} \Rightarrow y^* = \frac{\frac{a}{x_1} + b + \frac{a}{x_2} + b}{2} \Rightarrow y^* = \frac{a\left(\frac{1}{x_1} + \frac{1}{x_2}\right)}{2} + b$$

7. $y = bx^a$, by hypothesis the given function corresponds with geometric mean for both variables x and y:
8.

$$x^* = \sqrt{x_1 x_2} \Rightarrow y(x^*) = b(x_1 x_2)^{\frac{a}{2}}$$

$$y^* = \sqrt{y_1 y_2} \Rightarrow y^* = \sqrt{b{x_1}^a b{x_2}^a} \Rightarrow y^* = b\sqrt{{x_1}^a {x_2}^a} \Rightarrow y^* = b(x_1 x_2)^{\frac{a}{2}}$$

9. $y = \frac{x}{bx + a}$, by hypothesis the given function corresponds with harmonic mean for both variables x and y:

$$x^* = \frac{2}{\frac{1}{x_1} + \frac{1}{x_2}} \Rightarrow y(x^*) = \frac{\frac{2}{\frac{1}{x_1} + \frac{1}{x_2}}}{\frac{2b}{\frac{1}{x_1} + \frac{1}{x_2}} + a} \Rightarrow y(x^*) = \frac{2}{2b + a\left(\frac{1}{x_1} + \frac{1}{x_2}\right)}$$

$$y^* = \frac{2}{\frac{1}{y_1} + \frac{1}{y_2}} \Rightarrow y^* = \frac{2}{\frac{bx_1 + a}{x_1} + \frac{bx_2 + a}{x_2}} \Rightarrow y^* = \frac{2}{b + \frac{a}{x_1} + b + \frac{a}{x_2}} \Rightarrow y^* = \frac{2}{2b + a\left(\frac{1}{x_1} + \frac{1}{x_2}\right)}$$

10. $y = ba^{1/x}$, by hypothesis the given function corresponds with harmonic mean for x and geometric mean for y:

$$x^* = \frac{2}{\frac{1}{x_1} + \frac{1}{x_2}} \Rightarrow y(x^*) = ba^{\frac{\frac{1}{x_1} + \frac{1}{x_2}}{2}} \Rightarrow y(x^*) = ba^{\frac{x_1 + x_2}{2x_1 x_2}}$$

$$y^* = \sqrt{y_1 y_2} \Rightarrow y^* = \sqrt{ba^{\frac{1}{x_1}} ba^{\frac{1}{x_2}}} \Rightarrow y^* = b\sqrt{a^{\frac{1}{x_1}} a^{\frac{1}{x_2}}} \Rightarrow y^* = ba^{\frac{\frac{1}{x_1} + \frac{1}{x_2}}{2}}$$

11. $y=\frac{1}{a\ln x+b}$, by hypothesis the given function corresponds with geometric mean for x and harmonic mean for y:

$$x^*=\sqrt{x_1x_2}\Rightarrow y(x^*)=\frac{1}{a\ln\sqrt{x_1x_2}+b}$$

$$y^*=\frac{2}{\frac{1}{y_1}+\frac{1}{y_2}}\Rightarrow y^*=\frac{2}{a\ln x_1+b+a\ln x_2+b}\Rightarrow y^*=\frac{2}{a\ln x_1x_2+2b}\Rightarrow y^*=\frac{1}{b+a\ln\sqrt{x_1x_2}}$$

12. $y=ax^2+b$, by hypothesis the given function corresponds with quadratic mean for x and arithmetic mean for y:

$$x^*=\sqrt{\frac{x_1^2+x_2^2}{2}}\Rightarrow y(x^*)=a\left(\frac{x_1^2+x_2^2}{2}\right)+b$$

$$y^*=\frac{y_1+y_2}{2}\Rightarrow y^*=\frac{ax_1^2+b+ax_2^2+b}{2}\Rightarrow y^*=\frac{a\left(x_1^2+x_2^2\right)}{2}+b$$

13. $y=ba^{x^2}$, by hypothesis the given function corresponds with quadratic mean for x and geometric mean for y:

$$x^*=\sqrt{\frac{x_1^2+x_2^2}{2}}\Rightarrow y(x^*)=ba^{\frac{x_1^2+x_2^2}{2}}$$

$$y^*=\sqrt{y_1y_2}\Rightarrow y^*=\sqrt{ba^{x_1^2}ba^{x_2^2}}\Rightarrow y^*=b\sqrt{a^{x_1^2}a^{x_2^2}}\Rightarrow y^*=ba^{\frac{x_1^2+x_2^2}{2}}$$

14. $y=\frac{1}{(ax^2+b)}$, by hypothesis the given function corresponds with quadratic mean for x and harmonic mean for y:

$$x^*=\sqrt{\frac{x_1^2+x_2^2}{2}}\Rightarrow y(x^*)=\frac{1}{\left(a\left(\frac{x_1^2+x_2^2}{2}\right)+b\right)}$$

$$y^*=\frac{2}{\frac{1}{y_1}+\frac{1}{y_2}}\Rightarrow y^*=\frac{2}{ax_1^2+b+ax_2^2+b}\Rightarrow y^*=\frac{2}{a(x_1^2+x_2^2)+2b}\Rightarrow y^*=\frac{1}{b+a\left(\frac{x_1^2+x_2^2}{2}\right)}$$

15. $y=\sqrt{ax^2+b}$, by hypothesis the given function corresponds with quadratic mean for both variables x and y:

$$x^*=\sqrt{\frac{x_1^2+x_2^2}{2}}\Rightarrow y(x^*)=\sqrt{a\left(\frac{x_1^2+x_2^2}{2}\right)+b}$$

$$y^*=\sqrt{\frac{y_1^2+y_2^2}{2}}\Rightarrow y^*=\sqrt{\frac{ax_1^2+b+ax_2^2+b}{2}}\Rightarrow y^*=\sqrt{b+\frac{a(x_1^2+x_2^2)}{2}}$$

16. $y=\sqrt{ax+b}$, by hypothesis the given function corresponds with arithmetic mean for x and quadratic mean for y:

$$x^*=\frac{x_1+x_2}{2}\Rightarrow y(x^*)=\sqrt{a\left(\frac{x_1+x_2}{2}\right)+b}$$

$$y^*=\sqrt{\frac{y_1^2+y_2^2}{2}}\Rightarrow y^*=\sqrt{\frac{ax_1+b+ax_2+b}{2}}\Rightarrow y^*=\sqrt{b+\frac{a(x_1+x_2)}{2}}$$

17. $y=\sqrt{a\ln x+b}$, by hypothesis the given function corresponds with geometric mean for x and quadratic mean for y:

$$x^*=\sqrt{x_1x_2}\Rightarrow y(x^*)=\sqrt{a\ln\sqrt{x_1x_2}+b}=\sqrt{\frac{a\ln x_1x_2}{2}+b}$$

$$y^*=\sqrt{\frac{y_1^2+y_2^2}{2}}\Rightarrow y^*=\sqrt{\frac{a\ln x_1+b+a\ln x_2+b}{2}}\Rightarrow y^*=\sqrt{b+\frac{a\ln(x_1x_2)}{2}}$$

18. $y=\sqrt{\frac{a}{x}+b}$, by hypothesis the given function corresponds with harmonic mean for x and quadratic mean for y:

$$x^*=\frac{2}{\frac{1}{x_1}+\frac{1}{x_2}}\Rightarrow y(x^*)=\sqrt{\frac{a\left(\frac{1}{x_1}+\frac{1}{x_2}\right)}{2}+b}$$

$$y^*=\sqrt{\frac{y_1^2+y_2^2}{2}}\Rightarrow y^*=\sqrt{\frac{\frac{a}{x_1}+b+\frac{a}{x_2}+b}{2}}\Rightarrow y^*=\sqrt{b+\frac{a\left(\frac{1}{x_1}+\frac{1}{x_2}\right)}{2}}$$

19. $y=ax^3+b$, by hypothesis the given function corresponds with cubic mean for x and arithmetic mean for y:

$$x^*=\sqrt[3]{\frac{x_1^3+x_2^3}{2}}\Rightarrow y(x^*)=a\left(\frac{x_1^3+x_2^3}{2}\right)+b$$

$$y^*=\frac{y_1+y_2}{2}\Rightarrow y^*=\frac{ax_1^3+b+ax_2^3+b}{2}\Rightarrow y^*=\frac{a\left(x_1^3+x_2^3\right)}{2}+b$$

20. $y=ba^{x^3}$, by hypothesis the given function corresponds with cubic mean for x and geometric mean for y:

$$x^*=\sqrt[3]{\frac{x_1^3+x_2^3}{2}}\Rightarrow y(x^*)=ba^{\frac{x_1^3+x_2^3}{2}}$$

$$y^*=\sqrt{y_1y_2}\Rightarrow y^*=\sqrt{ba^{x_1^3}ba^{x_2^3}}\Rightarrow y^*=b\sqrt{a^{x_1^3}a^{x_2^3}}\Rightarrow y^*=ba^{\frac{x_1^3+x_2^3}{2}}$$

21. $y=\dfrac{1}{(ax^3+b)}$, by hypothesis the given function corresponds with cubic mean for x and harmonic mean for y:

$$x^*=\sqrt[3]{\frac{x_1^3+x_2^3}{2}}\Rightarrow y(x^*)=\frac{1}{\left(a\left(\frac{x_1^3+x_2^3}{2}\right)+b\right)}$$

$$y^*=\frac{2}{\frac{1}{y_1}+\frac{1}{y_2}}\Rightarrow y^*=\frac{2}{ax_1^3+b+ax_2^3+b}\Rightarrow y^*=\frac{2}{a(x_1^3+x_2^3)+2b}\Rightarrow y^*=\frac{1}{b+a\left(\frac{x_1^3+x_2^3}{2}\right)}$$

22. $y=\sqrt{ax^3+b}$, by hypothesis the given function corresponds with cubic mean for x and quadratic mean for y:

$$x^*=\sqrt[3]{\frac{x_1^3+x_2^3}{2}}\Rightarrow y(x^*)=\sqrt{a\left(\frac{x_1^3+x_2^3}{2}\right)+b}$$

$$y^*=\sqrt{\frac{y_1^2+y_2^2}{2}}\Rightarrow y^*=\sqrt{\frac{ax_1^3+b+ax_2^3+b}{2}}\Rightarrow y^*=\sqrt{b+\frac{a(x_1^3+x_2^3)}{2}}$$

As given formulas are the most widely used functional dependencies, it gives us the opportunity to find an empirical formula for almost all non-linear dependencies without human intervention, but relying entirely on the computational machine. Since this process is automated, the researches in the field of statistical modeling can be carried out much faster and more efficiently.

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