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SIMILAR TRANSFORMATION OF THE STURM -LIOUVILLE OPERATOR

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In this paper we will find similar transformation of the Sturm-Liouville operator in a Hilbert space.

Consider the Sturm-Liouville equation on the interval (0,1)

$$\hat{L}y = -y'' + q(x)y = f$$

where the function $q(x)$ real-valued and from Hilbert space $L_2(0,1)$. Solution of the homogeneous equation is $y(x) = C_1 c(x) + C_2 s(x)$, where C_1, C_2 arbitrary constants. The fundamental solutions of the homogenous equation are $c(x) = 1 + \int_0^x [\mathcal{K}(x,t) + \mathcal{K}(x,-t)]dt$

and $s(x) = x + \int_0^x [\mathcal{K}(x,t) - \mathcal{K}(x,-t)]tdt$, where the function $\mathcal{K}(x,t)$ is the solution of the following Goursat problem

$$\begin{cases} \mathcal{K}_{xx}" - \mathcal{K}_{yy}" = q(t)\mathcal{K}(x,t) \\ \mathcal{K}(x,-x) = 0 \\ \mathcal{K}(x,x) = \frac{1}{2}\int_0^x q(t)dt \end{cases}$$

Domain of the maximal and minimal operators are the following Sobolev spaces respectively $D(\hat{L}) = W_2^2(0,1)$ and $D(L_0) = \overset{0}{W_2^2}(0,1)$. As a domain of the fixed operator L we choose Dirichlet condition $D(L) = \{y \in W_2^2(0,1):\ \ y(0) = y(1) = 0\}$. Operator defined by $y = L_K^{-1}f = L^{-1}f + Kf$ for any bounded operator $K: L_2(0,1) \to ker\hat{L}$, is correct restriction of the maximal operator. Then correct restriction of the maximal operator L_K acts on the following domain

$$D(L_K) = \{y \in W_2^2(0,1):\ y(0) = \int_0^1 [-y"(t) + q(t)y(t)]\sigma_1(t)dt;$$

$$y(1) = c(1)y(0) + s(1)\int_0^1 [-y"(t) + q(t)y(t)]\sigma_2(t)dt\}$$

For any $\sigma_1(x), \sigma_2(x) \in L_2(0,1)$ we can uniquely define operator K in the following form

$$Kf = c(x)\int_0^1 f(t)\sigma_1(t)dt + s(x)\int_0^1 f(t)\sigma_2(t)dt\ , \forall f(x) \in L_2(0,1)$$

Operator KL bounded if and only if operator K satisfies the condition $R(K^*) \subset D(L^*)$. Adjoint operator of K is

$$K^*f = \sigma_1(x)\int_0^1 c(t)f(t)dt + \sigma_2(x)\int_0^1 s(t)f(t)dt\ .$$

If the functions $\sigma_1(x), \sigma_2(x) \in D(L^*) = D(L)$ then KL is bounded and have the following form

$$KLy = c(x)\int_0^1 [-y"(t) + q(t)y(t)]\sigma_1(t)dt + s(x)\int_0^1 [-y"(t) + q(t)y(t)]\sigma_2(t)dt$$

Using the fact that $\sigma_1(x), \sigma_2(x) \in W_2^2(0,1)$ and $\sigma_1(0) = \sigma_1(1) = \sigma_2(0) = \sigma_2(1) = 0$, and integrating by parts we get

$$KLy = c(x)\int_0^1 y(t)[-\sigma_1"(t) + q(t)\sigma_1(t)]dt + s(x)\int_0^1 y(t)[-\sigma_2"(t) + q(t)\sigma_2(t)]dt$$

The operator KL_K is also bounded and we have to find the form of KL_K.

$$KL_Ky = c(x)\int_0^1 [-y"(t) + q(t)y(t)]\sigma_1(t)dt + s(x)\int_0^1 [-y"(t) + q(t)y(t)]\sigma_2(t)dt$$

$$= c(x)\{-y'(1)\sigma_1(1) + y'(0)\sigma_1(0) + y(1)\sigma_1'(1) - y(0)\sigma_1'(0)$$

$$+ \int_0^1 y(t)[-\sigma_1"(t) + q(t)\sigma_1(t)]dt\} + s(x)\{-y'(1)\sigma_2(1) + y'(0)\sigma_2(0)$$

$$+ y(1)\sigma_2'(1) - y(0)\sigma_2'(0) + \int_0^1 y(t)[-\sigma_2"(t) + q(t)\sigma_2(t)]dt\}$$

$$KL_K y = c(x)[y(1)\sigma_1'^{(1)} - y(0)\sigma_1'(0) + \int_0^1 y(t)[-\sigma_1''(t) + q(t)\sigma_1(t)]dt] +$$

$$+s(x)[y(1)\sigma_2'(1) - y(0)\sigma_2'(0) + \int_0^1 y(t)[-\sigma_2''(t) + q(t)\sigma_2(t)]dt]$$

To define operator KL_K we have to find $y(0)$ and $y(1)$ on $D(L_K)$

$$y(0) = \int_0^1 [-y''(t) + q(t)y(t)]\sigma_1(t)dt =$$

$$= -y'(1)\sigma_1(1) + y'(0)\sigma_1(0) + y(1)\sigma_1'(1) - y(0)\sigma_1'(0) + \int_0^1 y(t)[-\sigma_1''(t) + q(t)\sigma_1(t)]dt$$

and

$$y(1) = c(1)y(0) + s(1)\int_0^1 [-y''(t) + q(t)y(t)]\sigma_2(t)dt =$$

$$= c(1)y(0) + s(1)[-y'(1)\sigma_2(1) + y'(0)\sigma_2(0) + y(1)\sigma_2'(1) - y(0)\sigma'_2(0)$$

$$+ \int_0^1 y(t)[-\sigma''_2(t) + q(t)\sigma_2(t)]dt]$$

This two equation can be written as linear equation system.

$$\begin{cases} [1 + \sigma'_1(0)]y(0) - \sigma'_1(1)y(1) = \int_0^1 y(t)[-\sigma_1''(t) + q(t)\sigma_1(t)]dt \\ [\sigma'_2(0)s(1) - c(1)]y(0) + [1 - \sigma'_2(1)s(1)]y(1) = \int_0^1 y(t)[-\sigma''_2(t) + q(t)\sigma_2(t)]dt] \end{cases}$$

By solving this linear equation system we get

$$y(0) = \frac{1 - s(1)\sigma'_2(1)}{\triangle}\int_0^1 y(t)[-\sigma_1''(t) + q(t)\sigma_1(t)]dt$$

$$- \frac{\sigma'_1(1)}{\triangle}\int_0^1 y(t)[-\sigma''_2(t) + q(t)\sigma_2(t)]dt]$$

$$y(1) = \frac{c(1) - s(1)\sigma'_2(0)}{\triangle}\int_0^1 y(t)[-\sigma_1''(t) + q(t)\sigma_1(t)]dt$$

$$+ \frac{1 + \sigma'_1(0)}{\triangle}\int_0^1 y(t)[-\sigma''_2(t) + q(t)\sigma_2(t)]dt]$$

where $\triangle = 1 + \sigma'_1(1)c(1) - \sigma'_2(1)s(1) + \sigma'_1(0) - \sigma'_1(1)\sigma'_2(0)s(1) - \sigma'_1(0)\sigma'_2(1)s(1)$.

By concluding all above calculations we find exact form of bounded operator KL_K.

$$KL_Ky = c(x)\left[\frac{2\sigma'_1(1)c(1) - 2\sigma'_1(1)\sigma'_2(0)s(1) + 1 - \sigma'_2(1)s(1)}{\triangle}\int_0^1 y(t)[-\sigma_1"(t) + q(t)\sigma_1(t)]dt + \frac{\sigma'_1(1) + 2\sigma'_1(0)\sigma'_1(1)}{\triangle}\int_0^1 y(t)[-\sigma"_2(t) + q(t)\sigma_2(t)]dt\right]$$
$$+ s(x)\left[\frac{\sigma'_2(1)c(1) - \sigma'_2(0)}{\triangle}\int_0^1 y(t)[-\sigma_1"(t) + q(t)\sigma_1(t)]dt + \frac{2\sigma'_1(0)\sigma'_2(1) - \sigma'_2(1) + \triangle}{\triangle}\int_0^1 y(t)[-\sigma"_2(t) + q(t)\sigma_2(t)]dt\right]$$

In the work(1), it was proven that if operator L_K densely defined on the Hilbert space and $R(K^*) \subset D(L^*) \cap D(L_K^*)$ then operator KL_K bounded on Hilbert space and correct operator A_K is similar to operator L_K on $D(A_K) = D(L)$, where $A_K = L - KL_KL$.
Then the operator A_K has the form

$$A_Ky = -y"(x) + q(x)y(x)$$
$$- c(x)\left[\frac{2\sigma'_1(1)c(1) - 2\sigma'_1(1)\sigma'_2(0)s(1) + 1 - \sigma'_2(1)s(1)}{\triangle}\int_0^1 Ly(t)L\sigma_1(t)dt + \frac{\sigma'_1(1) + 2\sigma'_1(0)\sigma'_1(1)}{\triangle}\int_0^1 Ly(t)L\sigma_2(t)dt\right]$$
$$+ s(x)\left[\frac{\sigma'_2(1)c(1) - \sigma'_2(0)}{\triangle}\int_0^1 Ly(t)L\sigma_1(t)dt + \frac{2\sigma'_1(0)\sigma'_2(1) - \sigma'_2(1) + \triangle}{\triangle}\int_0^1 Ly(t)L\sigma_2(t)dt\right]$$

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