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Imperfect fluid cosmological model in modified gravity

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In this article, we considered a bulk viscous fluid in the formalism of modified gravity in which the general form of the gravitational action is *f*(*R*, *T*), where *R* is the curvature scalar and *T* is the trace of the energy momentum tensor, within the framework of a flat FRW space time. The cosmological model is dominated by bulk viscous matter with its total bulk viscous coefficient expressed as a linear combination of the velocity and acceleration of the expansion of the universe in such a way that $\xi = \xi_0 + \xi_1 \frac{\dot{a}}{a} + \xi_2 \frac{\ddot{a}}{\dot{a}}$, where ξ_0, ξ_1 and ξ_2 are constants. We take $p = (\gamma - 1)\rho$, where $0 \le \gamma \le 2$, as the equation of state for a perfect fluid. The exact solutions to the corresponding field equations are obtained by assuming a particular model of the form of $f(R, T) = R + 2f(T)$, where $f(T) = \lambda T$, λ is constant. We studied four possible scenarios of the universe for different values of γ, namely $\gamma = 0$, $\gamma = \frac{2}{3}$, $\gamma = 1$ and $\gamma = \frac{4}{3}$, with positive and negative ranges of λ to observe the accelerated expansion history of the universe. Finally, a big-rip singularity is observed.

1. Introduction

From the type Ia supernova observations, it is clear that the present universe is dominated by dark energy which provides the dynamical mechanism for the accelerated expansion of the universe ([1–3]). This was confirmed by observations of the Cosmic Microwave Background Radiation (CMBR) [4], Large Scale Structure (LSS) [5], the Sloan Digital Sky Survey (SDSS) [6], the Wilkinson Microwave Anisotropy Probe (WMAP) [7] etc. The strength of this acceleration has been a remarkable question in recent years. Many models have been introduced to explain this current acceleration of the universe. Generally there are two approaches to describe the current acceleration of the universe: one is to propose to modify the energy momentum tensor $T_{\mu\nu}$ in Einstein's field equations. The second approach is to modify the geometry of the space time in Einstein's equations.

The simplest candidate for dark energy is the cosmological constant ($\wedge$), which is so called because its energy density is constant with respect to time and space. However, it suffers from the coincidence problem and the fine tuning problem [8]. So, as a result, dynamical dark energy models, such as quintessence ([9,10]), k-essence ([11,12]) and perfect fluid models (like the Chaplygin gas model) ([13,14]) were considered.

Recently, modified gravity has become one of the most popular candidates for understanding the idea of dark energy. In modified gravity, the origin of dark energy is identified as a modification of gravity. In the literature, a number of modified

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theories have been discussed to explain the early and late time expansion of the universe. In modified theories one modifies the laws of gravity so that the late time accelerated expansion of the universe is realized without recourse to an explicit dark energy matter component. One of the simplest modified gravity models is the so-called $f(R)$ gravity in which the 4-dimensional action is given by some general function $f(R)$ of the Ricci scalar R:

$$S = \frac{1}{2k^2}\int d^4x\sqrt{-g}f(R) + S_m(g_{\mu\nu}, \psi_m), \tag{1}$$

where $k^2 = 8\pi G$ and S_m is a matter action with matter field ψ_m. The matter field in S_m obeys standard conservation equations. $f(R)$ gravity was first introduced in [15]. Subsequently, several authors ([16–29]) investigated the $f(R)$ theory of gravity from different perspectives. However, there are many other modified theories of gravity that have been developed, like $f(T)$ gravity, Gauss–Bonnet theory, Lovelock gravity, Horva–Lifshitz gravity, scalar-tensor theories of gravity, braneworld models and so on ([30–37]).

Recently, in [38] another modification of general relativity, the so called $f(R, T)$ gravity, where the gravitational Lagrangian is given by an arbitrary function of the curvature scalar R and T, the trace of the energy momentum tensor, has been developed. The $f(R, T)$ gravity model depends on a source term, representing the variation of the matter stress-energy tensor with respect to the metric. The general expression for this source term is obtained as a function of the matter Lagrangian L_m. Therefore, the different choices of L_m would each generate a specific set of field equations. This modified theory possesses some interesting results which are relevant in theoretical cosmology and astrophysics. In most of the cosmological models, the matter part of the universe has been considered as a perfect fluid. In the context of inflation, many authors investigated the idea that the bulk viscous fluids are capable of providing for the acceleration of the universe ([39–41]). A bulk viscous fluid is the unique viscous effect capable of modifying the background dynamics in a homogeneous and isotropic universe. The matter behaves like a viscous fluid in an early stage of the universe, the neutrino decoupling phase. It has been known that a perfect fluid with bulk viscosity can produce an acceleration without the help of a cosmological constant or some scalar field. This idea was extended to explain the late time acceleration of the universe ([42–46]). The effects of an inhomogeneous equation of state of the universe, such as the phantom era, future singularity and crossing the phantom barrier were discussed by [47]. The effects of the viscosity terms depending on the Hubble parameter and its derivatives in the dark energy equation of state were discussed in [48]. Brevik et al. [49] proved, in particular, that a viscous fluid (or, equivalently, one with an inhomogeneous (imperfect) equation of state) is perfectly able to produce a Little Rip cosmology as a purely viscosity effect. Bulk viscosity is a quite favorable phenomenon, compatible with the symmetry requirements of the homogeneous and isotropic universe. The dark energy phenomenon as an effect of the bulk viscosity has been investigated in [51]. The mechanism for the formation of bulk viscosity by the decay of a dark matter particle is discussed in [50].

The motivation of the present work is to discuss the present acceleration of the universe with the help of an imperfect fluid within the framework of $f(R, T)$ gravity. In this paper, the authors studied a homogeneous and isotropic universe with bulk viscosity in $f(R, T)$ gravity and discussed the effects of bulk viscosity as an explanation of the early and late time acceleration of the universe. They analyzed the cosmic evolution of a bulk viscous matter dominated universe with the bulk viscous coefficient ξ depending on both the velocity and acceleration of the expanding universe as $\xi = \xi_0 + \xi_1\frac{\dot{a}}{a} + \xi_2\frac{\ddot{a}}{\dot{a}}$, where a is the scale factor of the universe and ξ_0, ξ_1 and ξ_2 are constants. The exact solutions of the field equations are obtained by assuming the simplest form of $f(R, T) = R + 2f(T)$, where $f(T) = \lambda T$.

2. Brief review of *f*(*R*, *T*) gravity and its field equations

The $f(R, T)$ theory is a modification of Einstein's general theory of relativity in which the Einstein–Hilbert Lagrangian, R, is replaced by an arbitrary function of the curvature scalar R and the trace T of the energy momentum tensor. The following modification of the Einstein theory was proposed in ([38]).

The action for the modified gravity takes the following form:

$$S = \frac{1}{16\pi}\int (f(R, T) + 16\pi L_m)\sqrt{-g}d^4x, \tag{2}$$

where g is the determinate of the metric tensor $g_{\mu\nu}$ and L_m is the matter Lagrangian density.

The energy momentum tensor $T_{\mu\nu}$, defined from the matter Lagrangian density L_m, is given by

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}, \tag{3}$$

and its trace T is defined by $T = g^{\mu\nu}T_{\mu\nu}$.

Assuming that the Lagrangian density L_m of matter depends only in the metric tensor $g_{\mu\nu}$, not on its derivatives, one can obtain

$$T_{\mu\nu} = g_{\mu\nu}L_m - 2\frac{\partial L_m}{\partial g^{\mu\nu}}. \tag{4}$$

The variation of the action (2) with respect to $g^{\mu\nu}$ gives

$$\delta S = \frac{1}{16\pi}\int\left[f_R(R, T)\delta R + f_T(R, T)\frac{\delta T}{\delta g^{\mu\nu}}\delta g^{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R, T)\delta g^{\mu\nu} + 16\pi\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\right]\sqrt{-g}d^4x, \tag{5}$$

where $f_R(R,T)=\frac{\partial f(R,T)}{\partial R}$ and $f_T(R,T)=\frac{\partial f(R,T)}{\partial T}$. The variation of the Ricci scalar can be obtained as

$$\delta R=\delta(g^{\mu\nu}R_{\mu\nu}) \\ =R_{\mu\nu}\delta g^{\mu\nu}+g^{\mu\nu}(\nabla_\lambda\delta\Gamma^\lambda_{\mu\nu}-\nabla_\nu\delta\Gamma^\lambda_{\mu\lambda}), \tag{6}$$

where ∇_λ is the covariant derivative with respect to the symmetric connection associated with the metric g. The variation of the Christoffel symbols yields

$$\delta\Gamma^\lambda_{\mu\nu}=\frac{1}{2}g^{\lambda\alpha}(\nabla_\mu\delta g_{\nu\alpha}+\nabla_\nu\delta g_{\alpha\mu}-\nabla_\alpha\delta g_{\mu\nu}), \tag{7}$$

and the variation of the Ricci scalar is

$$\delta R=R_{\mu\nu}\delta g^{\mu\nu}+g_{\mu\nu}\Box\delta g^{\mu\nu}-\nabla_\mu\nabla_\nu\delta g^{\mu\nu}. \tag{8}$$

Thus (5) reduces to

$$\delta S=\frac{1}{16\pi}\int\Big[f_R(R,T)R_{\mu\nu}\delta g^{\mu\nu}+f_R(R,T)g_{\mu\nu}\Box\delta g^{\mu\nu}-f_R(R,T)\nabla_\mu\nabla_\nu\delta g^{\mu\nu}+f_T(R,T)\frac{\delta(g^{\alpha\beta}T_{\alpha\beta})}{\delta g^{\mu\nu}}\delta g^{\mu\nu} \\ -\frac{1}{2}g_{\mu\nu}f(R,T)\delta g^{\mu\nu}+16\pi\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\Big]\sqrt{-g}d^4x. \tag{9}$$

The variation of T with respect to the metric tensor is

$$\frac{\delta(g^{\alpha\beta}T_{\alpha\beta})}{\delta g^{\mu\nu}}=T_{\mu\nu}+\Theta_{\mu\nu}, \tag{10}$$

where

$$\Theta_{\mu\nu}\equiv g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}. \tag{11}$$

After integrating the second and third terms in Eq. (9), we obtained the field equations of the $f(R, T)$ gravity model as

$$f_R(R,T)R_{\mu\nu}-\frac{1}{2}f(R,T)g_{\mu\nu}+(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)f_R(R,T) \\ =8\pi T_{\mu\nu}-f_T(R,T)T_{\mu\nu}-f_T(R,T)\Theta_{\mu\nu}, \tag{12}$$

where $T_{\mu\nu}$ is the standard matter energy momentum tensor derived from Eq. (4).

$$\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}=\frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}}L_m+g_{\alpha\beta}\frac{\partial L_m}{\partial g^{\mu\nu}}-2\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}} \\ =\frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}}L_m+\frac{1}{2}g_{\alpha\beta}g_{\mu\nu}L_m-\frac{1}{2}g_{\alpha\beta}T_{\mu\nu}-2\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}. \tag{13}$$

From the condition $g_{\alpha\sigma}g^{\sigma\beta}=\delta^\beta_\alpha$, we have

$$\frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}}=-g_{\alpha\sigma}g_{\beta\gamma}\delta^{\sigma\gamma}_{\mu\nu}, \tag{14}$$

where $\delta^{\sigma\gamma}_{\mu\nu}=\frac{\delta g^{\sigma\gamma}}{\delta g^{\mu\nu}}$ is the generalized Kronecker symbol. Therefore $\Theta_{\mu\nu}$ is

$$\Theta_{\mu\nu}=-2T_{\mu\nu}+g_{\mu\nu}L_m-2g^{\alpha\beta}\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}. \tag{15}$$

The contraction of Eq. (12) yields

$$f_R(R,T)R+3\Box f_R(R,T)-2f(R,T)=8\pi T-f_T(R,T)T-f_T(R,T)\Theta \tag{16}$$

where $\Theta=g^{\mu\nu}\Theta_{\mu\nu}$ and $\Box\equiv\nabla_\mu\nabla^\mu$ is the d'Alembert operator. From the Eqs. (12) and (16) we obtain

$$f_R(R,T)\Big(R_{\mu\nu}-\frac{1}{3}Rg_{\mu\nu}\Big)+\frac{1}{6}f(R,T)g_{\mu\nu}=8\pi\Big(T_{\mu\nu}-\frac{1}{3}Tg_{\mu\nu}\Big)-f_T(R,T)\Big(T_{\mu\nu}-\frac{1}{3}Tg_{\mu\nu}\Big) \\ -f_T(R,T)\Big(\Theta_{\mu\nu}-\frac{1}{3}\Theta g_{\mu\nu}\Big)+\nabla_\mu\nabla_\nu f_R(R,T). \tag{17}$$

If we assume that the matter of the universe is a perfect fluid, then the stress energy momentum tensor of the matter Lagrangian is given by

$$T_{\mu\nu}=(p+\rho)u_\mu u_\nu-pg_{\mu\nu}, \tag{18}$$

and the matter Lagrangian can be taken as $L_m = -p$. The four velocity vector in the co-moving co-ordinates system is $u^\mu = (1, 0, 0, 0)$, which satisfies the conditions $u_\mu u^\mu = 1$ and $u^\mu \nabla_\nu u_\mu = 0$. Here p and ρ are the pressure and energy density of the perfect fluid, respectively. With the use of Eq. (15), we obtain for the variation of the stress-energy of a perfect fluid

$$\Theta_{\mu\nu} = -2T_{\mu\nu} - pg_{\mu\nu}. \tag{19}$$

It is important to note that the field equations in $f(R, T)$ gravity also depend on the physical nature of the matter field through the tensor $\Theta_{\mu\nu}$. Therefore, the $f(R, T)$ theory depends on the nature of the matter source. Here, we can obtain several theoretical models for different choices of $f(R, T)$. Harko et al. considered three different explicit forms of $f(R, T)$ as

$$f(R,T) = \begin{cases} R + 2f(T), \\ f_1(R) + f_2(T), \\ f_1(R) + f_2(R)f_3(T). \end{cases} \tag{20}$$

Subsequently several authors ([52–62]) studied some cosmological models in $f(R, T)$ modified gravity for different choice of the form of $f(R, T)$.

In this paper, we consider the following form of $f(R, T)$:

$$f(R,T) = R + 2f(T), \tag{21}$$

i.e., the action is given by the Einstein-Hilbert one plus a function of T. The term $2f(T)$ in the gravitational action modifies the gravitational interaction between matter and the curvature scalar R. Using Eq. (21), one can re-write the gravitational field equations defined in (12) as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} - 2f'(T)(T_{\mu\nu} + \Theta_{\mu\nu}) + f(T)g_{\mu\nu}, \tag{22}$$

which is considered as the field equation of $f(R, T)$ gravity for the above particular form of $f(R, T)$. Here the prime stands for the derivative of $f(T)$ with respect to T.

In this paper, we consider that the source of gravitation is a combination of a perfect fluid and a bulk viscous fluid. Therefore, the energy momentum tensor takes the form

$$T_{\mu\nu} = (\rho + \bar{p})u_\mu u_\nu - \bar{p}g_{\mu\nu} \tag{23}$$

and

$$\bar{p} = p - 3\xi H, \tag{24}$$

where ρ is the energy density, ξ is the coefficient of bulk viscosity, $\bar{p}$ is the effective pressure and p is the proper pressure. Here $H = \frac{\dot{a}}{a}$ is the Hubble parameter, where an overdot stands for the derivative with respect to cosmic time t. Hence, the Lagrangian density may be chosen as $L_m = -\bar{p}$ and the tensor $\Theta_{\mu\nu}$ in (19) reduces to

$$\Theta_{\mu\nu} = -2T_{\mu\nu} - \bar{p}g_{\mu\nu}. \tag{25}$$

Now using the Eqs. (23) and (25), the field Eq. (22) for a bulk viscous fluid is given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} + 2f'(T)T_{\mu\nu} + (2\bar{p}f'(T) + f(T))g_{\mu\nu}. \tag{26}$$

We consider the field Eq. (26) with the particular choice of $f(T) = \lambda T$, where λ is a constant, and assume that the metric of the universe is given by the flat FRW metric:

$$ds^2 = dt^2 - a^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{27}$$

Then the gravitational field equations are given by

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\rho + 2\lambda(\rho + \bar{p}) + \lambda T, \tag{28}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -8\pi\bar{p} + \lambda T, \tag{29}$$

where $T = \rho - 3\bar{p}$. The equation of continuity is given by

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + \bar{p}) = 0. \tag{30}$$

In the following section we chose the equation of state and bulk viscosity coefficient and try to solve for H.

3. Exact solution of the field equations

The field Eqs. (28) to (30) (by substituting $H = \frac{\dot{a}}{a}$) become:

$$3H^2 = 8\pi\rho + 2\lambda(\rho + \bar{p}) + \lambda T, \tag{31}$$

$$2\dot{H} + 3H^2 = -8\pi\bar{p} + \lambda T, \tag{32}$$

and

$$\dot{\rho} + 3H(\rho + \bar{p}) = 0. \tag{33}$$

Subtracting Eq. (31) from Eq. (32) yields

$$2\dot{H} + (8\pi + 2\lambda)(p + \rho) - 3(8\pi + 2\lambda)\xi H = 0. \tag{34}$$

We can choose the equation of state in the following form:

$$p = (\gamma - 1)\rho, \tag{35}$$

where γ is a constant known as the EoS parameter lying in the range $0 \leq \gamma \leq 2$. We assume a general form of the bulk viscous coefficient [63]:

$$\begin{aligned} \xi &= \xi_0 + \xi_1\frac{\dot{a}}{a} + \xi_2\frac{\ddot{a}}{\dot{a}} \\ &= \xi_0 + \xi_1 H + \xi_2\left(\frac{\dot{H}}{H} + H\right), \end{aligned} \tag{36}$$

where $\xi_0, \xi_1,$ and ξ_2 are constants.

In this paper we consider that $\xi_0, \xi_1,$ and ξ_2 all are non-zero, so that the total bulk viscous parameter, $\xi = \xi_0 + \xi_1\frac{\dot{a}}{a} + \xi_2\frac{\ddot{a}}{\dot{a}}$, depends on both the velocity and acceleration of the expansion of the universe. Therefore, the linear combination is more general rather than restrictive. Using Eqs. (24), (35) and (36) in Eq. (31), we get

$$\rho = \frac{3H\left[(1 - \lambda(\xi_1 + \xi_2))H - \lambda\xi_2\frac{\dot{H}}{H} - \lambda\xi_0\right]}{8\pi + 4\lambda - \lambda\gamma}. \tag{37}$$

Using Eqs. (35) and (37) in Eq. (34), we have

$$\begin{aligned} &\left[2 - 3(8\pi + 2\lambda)\left(\frac{\gamma\lambda}{(8\pi + 4\lambda - \lambda\gamma)} + 1\right)\xi_2\right]\dot{H} - \left[\frac{3\lambda\xi_0\gamma(8\pi + 2\lambda)}{8\pi + 4\lambda - \lambda\gamma} + 3\xi_0(8\pi + 2\lambda)\right]H \\ &+ \left[\frac{(8\pi + 2\lambda)3\gamma}{(8\pi + 4\lambda - \lambda\gamma)}(1 - \lambda(\xi_1 + \xi_2)) - 3(8\pi + 2\lambda)(\xi_1 + \xi_2)\right]H^2 = 0. \end{aligned} \tag{38}$$

This implies that

$$\begin{aligned} \dot{H} = &\frac{\left[\frac{3\lambda\xi_0\gamma(8\pi+2\lambda)}{(8\pi+4\lambda-\lambda\gamma)} + 3\xi_0(8\pi + 2\lambda)\right]}{\left[2 - 3(8\pi + 2\lambda)\left(\frac{\gamma\lambda}{(8\pi+4\lambda-\lambda\gamma)} + 1\right)\xi_2\right]}H \\ &- \frac{\left[\frac{(8\pi+2\lambda)3\gamma}{8\pi+4\lambda-\lambda\gamma}(1 - \lambda(\xi_1 + \xi_2)) - 3(8\pi + 2\lambda)(\xi_1 + \xi_2)\right]}{\left[2 - 3(8\pi + 2\lambda)\left(\frac{\gamma\lambda}{(8\pi+4\lambda-\lambda\gamma)} + 1\right)\xi_2\right]}H^2. \end{aligned} \tag{39}$$

Eq. (39) is a form of the Bernouli differential equation; solving (39), we get

$$H = \frac{k_1 e^{k_1 t}}{k_2 e^{k_1 t} + k_3}, \tag{40}$$

where $k_1 = \frac{[\frac{3\lambda\xi_0\gamma(8\pi+2\lambda)}{(8\pi+4\lambda-\lambda\gamma)} + 3\xi_0(8\pi+2\lambda)]}{[2-3(8\pi+2\lambda)(\frac{\gamma\lambda}{(8\pi+4\lambda-\lambda\gamma)}+1)\xi_2]}$, $k_2 = \frac{[\frac{(8\pi+2\lambda)3\gamma}{8\pi+4\lambda-\lambda\gamma}(1-\lambda(\xi_1+\xi_2))-3(8\pi+2\lambda)(\xi_1+\xi_2)]}{[2-3(8\pi+2\lambda)\left(\frac{\gamma\lambda}{(8\pi+4\lambda-\lambda\gamma)}+1\right)\xi_2]}$ and $k_3 = k_1 c_1$, with c_1 being a constant of integration. Using $H = \frac{\dot{a}}{a}$, the scale factor a is given by

$$a = k_4(k_2 e^{k_1 t} + k_3)^{\frac{1}{k_2}}, \tag{41}$$

Table 1
For $\gamma = 0$ $(p+\rho = 0)$.

Range of c_1	Range of λ	Constraints on bulk coefficients ξ_0, ξ_1 and ξ_2	Variation of q	Variation of H	Evolution of the universe
$c_1 = 0$	for all λ	$\xi_0 > 0, \xi_2 > 0$	$q = -1$	negative throughout the evolution	contracting and accelerating exponentially
$c_1 = 0$	for all λ	$\xi_0 > 0$ $\xi_1 + \xi_2 < 0$	$q = -1$	positive throughout the evolution	expanding and accelerating exponentially
$c_1 > 0$	$\lambda > 0$	$\xi_0 > 0$, $0 < \xi_2 < \frac{1}{12\pi}$	$q < -1$ and $q \to -1$ as $t \to \infty$	positive to negative	expanding and accelerating super exponentially to expanding and contracting exponentially
$c_1 > 0$	$\lambda > 0$	$\xi_1 + \xi_2 < 0$, $0 < \xi_2 < \frac{1}{12\pi}$	$q < -1$ and $q \to -1$ as $t \to \infty$	positive throughout the evolution	expanding and accelerating super exponentially to expanding and accelerating exponentially
$c_1 > 0$	$\lambda > 0$	$\xi_0 > 0$, $\xi_2 > \frac{2}{3(8\pi+2\lambda)}$, $\xi_1 + \xi_2 > 0$	negative to positive	negative to positive	expanding and accelerating to expanding and decelerating
$c_1 > 0$	$\lambda > 0$	$\xi_1 + \xi_2 < 0$	negative to positive	positive throughout the evolution	expanding and accelerating super exponentially to expanding and decelerating
$c_1 > 0$	$-4\pi < \lambda < \frac{1}{3\xi_2} - 4\pi$	$\xi_0 > 0$, $\xi_2 \geq \frac{1}{12\pi}$	$q < -1$ and $q \to -1$ as $t \to \infty$	positive to negative	expanding and accelerating super exponentially to accelerating and contracting
$c_1 > 0$	$-4\pi < \lambda < \frac{1}{3\xi_2} - 4\pi$	$\xi_1 + \xi_2 < 0$, $\xi_2 \geq \frac{1}{12\pi}$	$q < -1$ and $q \to -1$ as $t \to \infty$	positive throughout the evolution	expanding and accelerating super exponentially to accelerating and expanding exponentially
$c_1 > 0$	$\lambda < -4\pi$	$\xi_0 > 0$, $\xi_2 > 0$	negative to positive	positive to zero	expanding and accelerating super exponentially to accelerating in the standard way
$c_1 < 0$	$\lambda > 0$	$\xi_0 > 0$, $\xi_2 > \frac{2}{3(8\pi+2\lambda)}$	$q < 0$	negative to zero	accelerating and contracting to accelerating in the standard way
$c_1 < 0$	$\lambda > 0$	$\xi_0 > 0$, $0 < \xi_2 < \frac{2}{3(8\pi+2\lambda)}$	positive to negative and $q \to -1$ as $t \to \infty$	negative throughout the evolution	decelerating and contracting to accelerating and contracting
$c_1 < 0$	$-4\pi < \lambda < 0$	$\xi_0 > 0$, $\xi_2 > \frac{2}{3(8\pi+2\lambda)}$	q is negative throughout theevolution	negative to zero	accelerating and contracting to accelerating in the standard way
$c_1 < 0$	$-4\pi < \lambda < 0$	$\xi_0 > 0$, $0 < \xi_2 < \frac{2}{3(8\pi+2\lambda)}$	q is positive to negative and $q \to -1$ as $t \to \infty$	negative throughout the evolution	decelerating and contracting to accelerating and contracting
$c_1 < 0$	$\lambda < -4\pi$	$\xi_0 > 0$, $\xi_2 > 0$	q is negative throughout the evolution	negative to zero	accelerating and contracting to accelerating in the standard way

where k_4 is a constant of integration. The energy density can be calculated as

$$\rho = \frac{3k_1 e^{k_1 t}}{k_2 e^{k_1 t} + k_3}\left[(1-\lambda(\xi_1+\xi_2))\frac{k_1 e^{k_1 t}}{k_2 e^{k_1 t} + k_3} - \lambda\xi_2 \frac{k_1^2 k_3}{(k_2 e^{k_1 t} + k_3)^2} - \lambda\xi_0\right]\frac{1}{(8\pi + 4\lambda - \lambda\gamma)}. \quad (42)$$

The deceleration parameter is given by

$$q = -1 - \frac{k_3}{e^{k_1 t}}, \quad (43)$$

which depends on the cosmic time t. It seems that the bulk viscous fluid also produces a time dependent deceleration parameter (q), which describe the transition phases of the universe along with the deceleration or acceleration of the universe. In Tables 1–4, we presented the variation of the deceleration parameter (q) and Hubble parameter (H) involved with the bulk viscous coefficients ξ_0, ξ_1 and ξ_2 for different ranges of λ and c_1 in different types of the universe for $\gamma = 0$, $\gamma = \frac{2}{3}$, $\gamma = 1$ and $\gamma = \frac{4}{3}$.

4. Discussion

The following observations are made from Table 1 (for $\gamma = 0$, i.e. $(p+\rho = 0)$):

The deceleration parameter $q = -1$ throughout the evolution and the Hubble parameter H is negative throughout the evolution for $c_1 = 0,\ \xi_0 > 0,\ \xi_2 > 0$ and there are no restrictions for ξ_1 and λ, so that the universe is contracting and accelerating exponentially for $c_1 = 0, \xi_0 > 0,\ \xi_1 + \xi_2 < 0$ and for all values of λ. The deceleration parameter $q = -1$ and the Hubble parameter H is positive throughout the evolution for $c_1 = 0,\ \xi_0 > 0,\ \xi_1 + \xi_2 < 0$ and for all values of λ, so that the universe is expanding and accelerating exponentially for $c_1 = 0,\ \xi_0 > 0,\ \xi_1 + \xi_2 < 0$ and for all values of λ. For $c_1 > 0,\ \lambda > 0,\ -4\pi < \lambda < \frac{1}{3\xi_2} - 4\pi,\ \xi_0 > 0,\ \xi_2 \geq \frac{1}{12\pi}$ and $0 < \xi_2 < \frac{1}{12\pi}$, the deceleration parameter $q < -1$ for small t or the present time and $q \to -1$ for large t, i.e. $t \to \infty$, and the Hubble parameter H varies from positive to negative as time

Table 2
For $\gamma = \frac{2}{3}$ $(p + \frac{\rho}{3} = 0)$.

Range of c_1	Range of λ	Constraints on bulk coefficients ξ_0, ξ_1 and ξ_2	Variation of q	Variation of H	Evolution of the universe
$c_1 = 0$	for all λ	$\xi_0 > 0, \xi_1 > 0, \xi_2 > 0$	$q = -1$	negative to positive for some λ and positive to negative for some λ	accelerating and contracting to accelerating and expanding exponentially and vice versa
$c_1 > 0$	$\lambda > -2.4\pi$	$\xi_0 > 0, \xi_1 > 0, \xi_2 < \frac{1}{2(\lambda+4\pi)(\lambda+2\pi)(24\pi+10\lambda)}, \xi_2 > 0$	negative and $q \to -1$ as $t \to \infty$	positive throughout the evolution	expanding and accelerating super exponentially to expanding and accelerating exponentially
$c_1 > 0$	$-4\pi < \lambda < -2.4\pi$	$\xi_0 > 0, \xi_1 > 0, \xi_2 < \frac{1}{2(\lambda+4\pi)(\lambda+2\pi)(24\pi+10\lambda)}$ $\xi_2 > 0$	negative to positive	negative to zero	accelerating and contracting
$c_1 > 0$	$\lambda < -4\pi$	$\xi_0 > 0, \xi_1 > 0, \xi_2 > 0$	$q < -1$ throughout the evolution	positive throughout the evolution	expanding and accelerating super exponentially
$c_1 < 0$	$\lambda > -2.4\pi$	$\xi_0 > 0, \xi_1 > 0, \xi_2 < \frac{1}{2(\lambda+4\pi)(\lambda+2\pi)(24\pi+10\lambda)}, \xi_2 > 0$	positive and $q \to -1$ as $t \to \infty$	positive throughout the evolution	decelerating and expanding to accelerating and expanding exponentially
$c_1 < 0$	$-4\pi < \lambda < -2.4\pi$	$\xi_0 > 0, \xi_1 > 0, \xi_2 < \frac{1}{2(\lambda+4\pi)(\lambda+2\pi)(24\pi+10\lambda)}, \xi_2 > 0$	positive throughout the evolution	negative to zero	decelerating and contracting
$c_1 < 0$	$\lambda < -4\pi$	$\xi_0 > 0, \xi_1 > 0, \xi_2 > 0$	positive to $q < -1$ for large t	positive throughout the evolution	decelerating and expanding to accelerating super exponentially

Table 3
For $\gamma = 1$ $(p = 0)$.

Range of c_1	Range of λ	Constraints on bulk coefficients ξ_0, ξ_1 and ξ_2	Variation of q	Variation of H	Evolution of the universe
$c_1 = 0$	for all λ	$\xi_0 > 0, 0 \le \xi_2 < \frac{2(8\pi+3\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$ $\xi_1 + \xi_2 \to 0$ or $\xi_1 + \xi_2 \le 0$	$q = -1$ throughout the evolution	positive	accelerating and expanding exponentially
$c_1 > 0$	$\lambda \le -4\pi,\ \lambda \ge 0$	$\xi_0 > 0, 0 \le \xi_2 < \frac{2(8\pi+3\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$ $\xi_1 + \xi_2 \to 0$ or $\xi_1 + \xi_2 \le 0$	$q < -1$ to $q = -1$ as $t \to \infty$	positive	accelerating and expanding super exponentially to expanding and accelerating exponentially
$c_1 > 0$	$-4\pi < \lambda < 0$	$\xi_0 > 0, 0 \le \xi_2 < \frac{2(8\pi+3\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$ $\xi_1 + \xi_2 \to 0$ or $\xi_1 + \xi_2 \le 0$	$q < -1$	positive	accelerating and expanding super exponentially
$c_1 < 0$	$\lambda \le -4\pi,\ \lambda \ge 0$	$\xi_0 > 0, 0 \le \xi_2 < \frac{2(8\pi+3\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$ $\xi_1 + \xi_2 \to 0$ or $\xi_1 + \xi_2 \le 0$	negative to $q = -1$ as $t \to \infty$	positive	accelerating and expanding in the standard the way to expanding and accelerating exponentially
$c_1 < 0$	$-4\pi < \lambda < 0$	$\xi_0 > 0, 0 \le \xi_2 < \frac{2(8\pi+3\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$ $\xi_1 + \xi_2 \to 0$ or $\xi_1 + \xi_2 \le 0$	negative to positive	positive	accelerating and expanding in the standard way to decelerating in the standard way

increases, so the universe is expanding and accelerating super exponentially to contracting exponentially. The universe is expanding and accelerating super exponentially to expanding and accelerating exponentially as the deceleration parameter $q < -1$ for the present time and $q \to -1$ as $t \to \infty$, and the Hubble parameter is positive throughout the evolution for $c_1 > 0,\ \lambda > 0,\ \xi_1 + \xi_2 < 0,\ 0 < \xi_2 < \frac{1}{12\pi},\ -4\pi < \lambda < \frac{1}{3\xi_2} - 4\pi$ and $\xi_2 \ge \frac{1}{12\pi}$. The universe is expanding and accelerating to expanding and decelerating as the deceleration parameter q and the Hubble parameter H varies from negative to positive for $c_1 > 0,\ \lambda > 0,\ \xi_0 > 0,\ \xi_2 > \frac{2}{3(8\pi+2\lambda)}$ and $\xi_1 + \xi_2 > 0$. The universe is expanding and accelerating super exponentially to expanding and decelerating in the standard way as the deceleration parameter q varies from negative to positive, and the Hubble parameter H is positive throughout the evolution for $c_1 > 0, \lambda > 0$ and $\xi_1 + \xi_2 < 0$. For $c_1 > 0,\ \lambda < -4\pi,\ \xi_0 > 0$ and $\xi_2 > 0$, the deceleration parameter varies from negative to positive and the Hubble parameter varies from positive to zero, so the universe is expanding and accelerating super exponentially to accelerating in the standard way. The universe is accelerating and contracting to accelerating in the standard way as the deceleration parameter is negative throughout the evolution and the Hubble parameter varies from negative to zero for $c_1 < 0,\ \lambda > 0,\ \xi_0 > 0,\ \xi_2 > \frac{2}{3(8\pi+2\lambda)},\ -4\pi < \lambda < 0,$ $\lambda < -4\pi$ and $\xi_2 > 0$. The universe is decelerating and contracting to accelerating and contracting in the standard way as the deceleration parameter varies from positive to negative and $q \to -1$ as $t \to \infty$, and the Hubble parameter is negative throughout the evolution for $c_1 < 0,\ \lambda > 0,\ \xi_0 > 0,\ 0 < \xi_2 < \frac{2}{3(8\pi+2\lambda)},\ -4\pi < \lambda < 0$.

The following observations are made from Table 2 (for $\gamma = \frac{2}{3}$, i.e. $(p + \frac{\rho}{3} = 0)$): The universe is accelerating and contracting to accelerating and expanding exponentially and vice versa as the deceleration parameter $q = -1$ and the Hubble parameter H varies from negative to positive for some λ and positive to negative for some λ, when $c_1 = 0,\ \xi_0 > 0,\ \xi_1 > 0,\ \xi_2 > 0$

Table 4
For $\gamma = \frac{4}{3}$ $(p = \frac{\rho}{3})$.

Range of c_1	Range of λ	Constraints on bulk coefficients ξ_0, ξ_1 and ξ_2	Variation of q	Variation of H	Evolution of the universe
$c_1 = 0$	for all λ	$\xi_0 > 0$, $0 \le \xi_2 < \frac{2(24\pi+8\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$ $\xi_1+\xi_2 \to 0$ or $\xi_1+\xi_2 \le 0$	$q = -1$ throughout the evolution	positive	accelerating and expanding exponentially
$c_1 > 0$	$\lambda > -3\pi$	$\xi_0 > 0$, $\xi_1 > 0$ $0 \le \xi_2 < \frac{2(24\pi+8\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$ $\xi_1+\xi_2 \to 0$ or $\xi_1+\xi_2 \le 0$	$q < -1$ to $q = -1$ as $t \to \infty$	positive	accelerating and expanding super exponentially to expanding and accelerating exponentially
$c_1 > 0$	$-4\pi < \lambda < -3\pi$	$\xi_0 > 0$, $\xi_1 > 0$ $0 \le \xi_2 < \frac{2(24\pi+8\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$ $\xi_1+\xi_2 \to 0$ or $\xi_1+\xi_2 \le 0$	$q = -1$ to $q < -1$	positive to zero	accelerating and expanding exponentially to accelerating and expanding super exponentially
$c_1 > 0$	$\lambda < -4\pi$	$\xi_0 > 0$, $\xi_1 > 0$ $\xi_2 > 0$	$q = -1$ to $q < -1$ as $t \to \infty$	positive to zero	accelerating and expanding exponentially to accelerating and expanding super exponentially
$c_1 < 0$	$\lambda > -3\pi$	$\xi_0 > 0$, $\xi_1 > 0$ $0 \le \xi_2 < \frac{2(24\pi+8\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$ $\xi_1+\xi_2 \to 0$ or $\xi_1+\xi_2 \le 0$	$-1 < q < 0$ to $q = -1$ as $t \to \infty$	positive	accelerating and expanding in the standard way to expanding and accelerating exponentially
$c_1 < 0$	$-4\pi < \lambda < -3\pi$	$\xi_0 > 0$, $\xi_1 > 0$ $0 \le \xi_2 < \frac{2(24\pi+8\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$ $\xi_1+\xi_2 \to 0$ or $\xi_1+\xi_2 \le 0$	$-1 < q < 0$ to $q > 0$	positive to zero	accelerating and expanding in the standard way to decelerating and expanding
$c_1 < 0$	$\lambda < -4\pi$	$\xi_0 > 0$, $\xi_1 > 0$ $\xi_2 > 0$	$q = -1$ to $q > 0$ as $t \to \infty$	positive to zero	accelerating and expanding exponentially to decelerating in the standard way

and for all λ. When $c_1 > 0,\ \lambda > -2.4\pi,\ \xi_0 > 0,\ \xi_1 > 0$ and $0 < \xi_2 < \frac{1}{2(\lambda+4\pi)(\lambda+2\pi)(24\pi+10\lambda)}$, the deceleration parameter q is negative and $q \to -1$ as $t \to \infty$, and the Hubble parameter H is positive throughout the evolution, so the universe is expanding and accelerating super exponentially to expanding and accelerating exponentially. When $c_1 > 0,\ -4\pi < \lambda < -2.4\pi,\ \xi_0 > 0,\ \xi_1 > 0$ and $0 < \xi_2 < \frac{1}{2(\lambda+4\pi)(\lambda+2\pi)(24\pi+10\lambda)}$, the deceleration parameter varies from negative to positive and the Hubble parameter varies from negative to zero, so the universe is accelerating and contracting in the standard way. When $c_1 > 0,\ \lambda < -4\pi,\ \xi_0 > 0,\ \xi_1 > 0$ and $\xi_2 > 0$, the deceleration parameter $q < -1$ throughout the evolution and the Hubble parameter H is positive throughout the evolution, so the universe is expanding and accelerating super exponentially. When $c_1 < 0,\ \lambda > -2.4\pi,\ \xi_0 > 0,\ \xi_1 > 0$ and $0 < \xi_2 < \frac{1}{2(\lambda+4\pi)(\lambda+2\pi)(24\pi+10\lambda)}$, the deceleration parameter q varies from positive to -1 as $t \to \infty$ and the Hubble parameter H is positive throughout the evolution, so the universe is decelerating and expanding to accelerating and expanding exponentially. When $c_1 < 0,\ -4\pi < \lambda < -2.4\pi,\ \xi_0 > 0,\ \xi_1 > 0$ and $0 < \xi_2 < \frac{1}{2(\lambda+4\pi)(\lambda+2\pi)(24\pi+10\lambda)}$, the deceleration parameter is positive throughout the evolution and the Hubble parameter varies from negative to zero, so the universe is decelerating and contracting in the standard way. When $c_1 < 0,\ \lambda < -4\pi,\ \xi_0 > 0,\ \xi_1 > 0$ and $\xi_2 > 0$, the deceleration parameter q varies from positive to $q < -1$ for large t and the Hubble parameter H is positive throughout the evolution, so the universe is decelerating and expanding to accelerating super exponentially.

The following observations are made from Table 3 (for $\gamma = 1$, i.e. $(p = 0)$): The universe is accelerating and expanding exponentially as the deceleration parameter $q = -1$ and the Hubble parameter is positive throughout the evolution for $c_1 = 0,\ \xi_0 > 0,\ \xi_1+\xi_2 \to 0$ or $\xi_1+\xi_2 \le 0$ and $0 \le \xi_2 < \frac{2(8\pi+3\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$. The universe is accelerating and expanding super exponentially to expanding and accelerating exponentially as the deceleration parameter q varies from $q < -1$ to $q = -1$ as $t \to \infty$ and the Hubble parameter is positive for $c_1 > 0,\ \lambda \ge 0,\ \lambda \le -4\pi,\ \xi_0 > 0,\ \xi_1+\xi_2 \to 0$ or $\xi_1+\xi_2 \le 0$ and $0 \le \xi_2 < \frac{2(8\pi+3\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$. When $c_1 > 0,\ -4\pi < \lambda < 0,\ \xi_0 > 0,\ \xi_1+\xi_2 \to 0$ or $\xi_1+\xi_2 \le 0$ and $0 \le \xi_2 < \frac{2(8\pi+3\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$, the deceleration parameter $q < -1$ and the Hubble parameter is positive, so the universe is accelerating and expanding super exponentially. When $c_1 < 0,\ \lambda \ge 0,\ \lambda \le -4\pi, \xi_0 > 0, \xi_1+\xi_2 \to 0$ or $\xi_1+\xi_2 \le 0$ and $0 \le \xi_2 < \frac{2(8\pi+3\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$, the deceleration parameter varies from negative to $q = -1$ as $t \to \infty$ and the Hubble parameter is positive, so the universe is accelerating and expanding in the standard way to expanding and accelerating exponentially. When $c_1 < 0,\ -4\pi < \lambda < 0,\ \xi_0 > 0,\ \xi_1+\xi_2 \to 0$ or $\xi_1+\xi_2 \le 0$ and $0 \le \xi_2 < \frac{2(8\pi+3\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$, the deceleration parameter varies from negative to positive and the Hubble parameter is positive, so the universe is accelerating and expanding in the standard way to decelerating in the standard way.

The following observations are made from Table 4 (for $\gamma = \frac{4}{3}$, i.e. $(p = \frac{\rho}{3})$): The universe is accelerating and expanding exponentially as the deceleration parameter $q = -1$ and the Hubble parameter H is positive throughout the evolution for $c_1 = 0,\ \xi_0 > 0,\ \xi_1+\xi_2 \to 0$ or $\xi_1+\xi_2 \le 0$ and $0 \le \xi_2 < \frac{2(24\pi+8\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$. The universe is accelerating and expanding super exponentially to expanding and accelerating exponentially as the deceleration parameter varies from $q < -1$ to $q = -1$ as $t \to \infty$ and the Hubble parameter is positive for $c_1 > 0,\ \lambda > -3\pi,\ \xi_0 > 0,\ \xi_1 > 0,\ \xi_1+\xi_2 \to 0$ or $\xi_1+\xi_2 \le 0$ and $0 \le \xi_2 < \frac{2(24\pi+8\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$. When $c_1 > 0,\ -4\pi < \lambda < -3\pi,\ \xi_0 > 0,\ \xi_1 > 0,\ \xi_1+\xi_2 \to 0$ or $\xi_1+\xi_2 \le 0, 0 \le \xi_2 < \frac{2(24\pi+8\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$ and $\lambda < -4\pi, \xi_0 > 0, \xi_1 > 0, \xi_2 > 0$, the deceleration parameter varies from $q = -1$ to $q < -1$ as $t \to \infty$ and the Hubble parameter varies from positive to zero, so the universe is accelerating and expanding exponentially to accelerating and expanding super exponentially. When $c_1 < 0,\ \lambda > -3\pi,\ \xi_0 > 0,\ \xi_1 > 0,\ \xi_1+\xi_2 \to 0$ or $\xi_1+\xi_2 \le 0$ and $0 \le \xi_2 < \frac{2(24\pi+8\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$, the deceleration parameter varies from $-1 < q < 0$ to $q = -1$ as $t \to \infty$ and the Hubble parameter is positive, so the universe is accelerating and expanding in the standard way to expanding and accelerating exponentially. When $c_1 < 0,\ -4\pi < \lambda < -3\pi,\ \xi_0 > 0,\ \xi_1 > 0,\ \xi_1 + \xi_2 \to 0$ or $\xi_1 + \xi_2 \leq 0$ and $0 \leq \xi_2 < \frac{2(24\pi+8\lambda)}{3(8\pi+2\lambda)(8\pi+4\lambda)}$, the deceleration parameter varies from $-1 < q < 0$ to $q > 0$ as $t \to \infty$ and the Hubble parameter varies from positive to zero, so the universe is accelerating and expanding in the standard way to decelerating and expanding. When $c_1 < 0,\ \lambda < -4\pi,\ \xi_0 > 0,\ \xi_1 > 0,\ \xi_2 > 0$, the deceleration parameter $q = -1$ to $q > 0$ as $t \to \infty$ and the Hubble parameter varies from positive to zero, so the universe is accelerating and expanding exponentially to decelerating in the standard way.

5. Conclusions and further comments

In this article, we carried out a study of the bulk viscous matter dominated universe with the bulk viscosity coefficient $\xi = \xi_0 + \xi_1 \frac{\dot{a}}{a} + \xi_2 \frac{\ddot{a}}{\dot{a}}$ within the framework of $f(R, T)$ gravity. In this work, the model proposed by Avelino and Nucamendi [64] has been extended and improved upon to reflect a more general situation. We extend their work into $f(R, T)$ gravity with the bulk viscous coefficient being proportional to a linear combination of three terms, $\xi = \xi_0 + \xi_1 \frac{\dot{a}}{a} + \xi_2 \frac{\ddot{a}}{\dot{a}}$, (where ξ_0, ξ_1 and ξ_2 are the positive constants) rather than $\xi = \xi_0 + \xi_1 \frac{\dot{a}}{a}$. The bulk viscous matter simultaneously represents dark matter and dark energy and causes the recent acceleration, therefore this model solves the coincidence problem automatically. The bulk viscous fluid is a viable candidate for explaining the early and late time expansion of the universe. Therefore, in this article we explored the evolution of the universe driven by a kind of viscous fluid by assuming a general form of the bulk viscous coefficients. We discussed the expansion history of the universe with viscosity. We discussed the various phases and their possible transitions for all possible ranges of λ, c_1 with ξ_0, ξ_1 and ξ_2. We obtained the time dependent deceleration parameter q and the Hubble parameter H which describe the decelerated/accelerated evolution and the transition from a decelerated/accelerated to an accelerated/decelerated phase. The existence of sudden singularities in the Friedmann universes of higher-order Lagrangian theories of gravity has been discussed in [65]. The big-rip singularity is called the type I singularity [66], the behavior of a type I singularity is as follows: $a \to \infty, \rho \to \infty, |p| \to \infty$, as $t \to t_s$. Finally, we observed a big-rip or type I singularity at $t = \frac{[2-3(8\pi+2\lambda)\left(\frac{\gamma\lambda}{(8\pi+4\lambda-\lambda\gamma)}+1\right)\xi_2]}{[\frac{3\lambda\xi_0\gamma(8\pi+2\lambda)}{(8\pi+4\lambda-\lambda\gamma)}+3\xi_0(8\pi+2\lambda)]} \times \ln[c_1 \frac{[\frac{3\lambda\xi_0\gamma(8\pi+2\lambda)}{(8\pi+4\lambda-\lambda\gamma)}+3\xi_0(8\pi+2\lambda)]}{[\frac{(8\pi+2\lambda)3\gamma}{8\pi+4\lambda-\lambda\gamma}(1-\lambda(\xi_1+\xi_2))-3(8\pi+2\lambda)(\xi_1+\xi_2)]}]$ in our model. In conclusion, the authors strongly emphasize that a perfect fluid is just a limiting case of a general bulk viscous medium which is more practical in the astrophysical sense. Therefore, it is meaningful to study the early and late time cosmic evolution of the universe with a bulk viscous fluid in $f(R, T)$ gravity, which describes the evolution of the universe in various ways for different ranges of λ, ξ_0, ξ_1 and ξ_2. Subsequently, our paper provides an interesting topic for the further study of such kinds of important cosmological models, where the matter content of the universe is filled with a bulk viscous fluid for variable bulk viscous coefficients, and, at the same time, the following problems can be considered in future research work.

1. The same work may be extended to Kaluza–Klein theory.
2. In this paper, the authors have taken $f(R, T) = R + 2f(T)$, where $f(T) = \lambda T$. Researchers may think of different forms of $f(R, T)$ and $f(T)$, say instead of linear, they may consider a quadratic form.
3. We may look towards the validity of the second law of thermodynamics in the presence of the bulk viscous fluid with variable coefficients.
4. In this paper, the authors have taken the matter part of the universe to be filled with an imperfect fluid within the framework of isotropic space time; researchers may extend this work within the framework of an anisotropic space time (viz. a Bianchi type space time).
5. Double special relativity has been generalized to curved space-time, and this doubly general theory of relativity is called gravity's rainbow [67]. In this theory, the geometry of space-time depends on the energy of the test particle. Therefore, the geometry of space-time is represented by a family of energy dependent metrics, forming a rainbow of metrics. This is the reason why the theory is called gravity's rainbow. The gravity's rainbow theory has been extensively studied in order to explore various aspects for black holes and cosmology ([68–85]). Therefore, it would be interesting to study our work in gravity's rainbow. Subsequently, our work may also be extended into the generalized uncertainty principle (GUP) ([86–90]). Third quantization has been discussed by many authors ([91–96]). It may be noted that it is possible to study this third quantization deformation of the Wheeler–DeWitt equation for $f(R, T)$ modified gravity.

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